Data

Euclid

Data

Translated by Robert Simson, 1830

PREFACE.

Euclid's data is the first in order of the books written by the antient Geometers to facilitate and promote the method of Resolution or Analysis. In the general, a thing is said to be given which is either actually exhibited, or can be found out, that is, which is either known by Hypothesis, or that can be demonstrated to be known; and the Propositions in the Book of Euclid's Data shew what things can be found out or known from those that by Hypothesis are already known; so that in the Analysis or Investigation of a Problem, from the things that are laid down to be known or given, by the help of these Propositions other things are demonstrated to be given, and from these other things are again shewn to be given, and so on, until that which was proposed to be found out in the Problem is demonstrated to be given, and when this is done the Problem is solved, and its Composition is made and derived from the Compositions of the Data which were made use of in the Analysis. And thus the Data of Euclid are of the most general and necessary use in the solution of Problems of every kind.

Euclid is reckoned to be the Author of the Book of the Data both by the antient and modern Geometers; and there seems to be no doubt of his having written a Book on this subject, but which in the course of so many ages has been much vitiated by unskilful Editors in several places, both in the order of the Propositions, and in the Definitions and Demonstrations themselves. To correct the errors which are now found in it, and bring it nearer to the accuracy with which it was, no doubt, at first written by Euclid, is the design of this Edition, that so it may be rendered more useful to Geometers, at least to beginners who desire to learn the investigatory method of the antients. And for their sakes the Composition of most of the Data are subjoined to their Demonstrations, that the Compositions of Problems solved by help of the

Data may be the more easily made.

Marinus the Philosopher's preface which in the Greek Edition is prefixed to the Data is here left out, as being of no use to understand them. at the end of it he says that Euclid has not used the synthetical, but the analytical method in delivering them; in which he is quite mistaken; for in the Analysis of a Theorem the thing to be demonstrated is assumed in the Analysis; but in the Demonstrations of the Data, the thing to be demonstrated, which is that something or other is given, is never once assumed in the Demonstration, from which it is manifest that every one of then is demonstrated synthetically; tho' indeed if a Proposition of the Data be turned into a Problem, for example the 84th or 85th in the former Editions, which here are the 85th and 86th, the Demonstration of the Proposition becomes the Analysis of the Problem.

Wherein this Edition differs from the Greek, and the reasons of the alterations from it will be shewn in the Notes at the end of the Data.

DEFINITIONS.

I.

Spaces, lines and angles are said to be given in magnitude, when equals to them can be found.

II.

A ratio is said to be given, when a ratio of a given magnitude to a given magnitude which is the same ratio with it can be found.

III

Rectilineal figures are said to be given in species, which have each of their angles given, and the ratios of their sides given.

IV.

Points, lines and spaces are said to be given in position, which have always the same situation, and which are either actually exhibited, or can be found.

A.

An angle is said to be given in position, which is contained by straight lines given in position.

V.

A circle is said to be given in magnitude, when a straight line from its center to the circumference is given in magnitude.

VI.

A circle is said to be given in position and magnitude, the center of which is given in position, and a straight line from it to the circumference is given in magnitude.

VII.

Segments of circles are said to be given in magnitude, when the angles in them, and their bases are given in magnitude.

VIII.

Segments of circles are said to be given in position and magnitude, when the angles in them are given in magnitude, and their bases are given both in position and magnitude.

IX.

A magnitude is said to be greater than another by a given magnitude, when this given magnitude being taken from it, the remainder is equal to the other magnitude.

X.

A magnitude is said to be less than another by a given magnitude, when this given magnitude being added to it, the whole is equal to the other magnitude.

PROPOSITIONS.

PROPOSITION I. THE ratios of given magnitudes to one another is given.

Let A, B be two given magnitudes, the ratio of A to B is given. Because A is a given magnitude, there may be found one equal to it; let this be C. and because B is given, one equal to it may be found; let it be D. and since A is equal to C, and B to D; therefore A is to B, as C to D; and consequently the ratio of A to B is given, because the ratio of the given magnitudes C, D which is the same with it has been found.

PROP. II.

2. IF a given magnitude has a given ratio to another magnitude, “and if unto the two magnitudes by which the given ratio is exhibited, and the given magnitude, a fourth proportional can be found;” the other magnitude is given.

Let the given magnitude A have a given ratio to the magnitude B; if a fourth proportional can be found to the three magnitudes above-named, B is given in magnitude.

Because A is given, a magnitude may be found equal to it; let this be C. and because the ratio of A to B is given, a ratio which is the same with it may be found; let this be the ratio of the given magnitude E to the given magnitude F. unto the magnitudes E, F, C find a fourth proportional D, which, by the Hypothesis, can be done. wherefore because A is to B, as E to F; and as E to F,

so is C to D; A is to B, as C to D. but A is equal to C, therefore B is equal to D. the magnitude B is therefore given, because a magnitude D equal to it has been found.

The limitation within the inverted commas is not in the Greek text, but is now necessarily added; and the same must be understood in all the Propositions of the Book which depend upon this second Proposition, where it is not expressly mentioned. See the Note upon it.

PROP. III.

3. IF any given magnitudes be added together, their sum shall be given.

Let any given magnitudes AB, BC be added together, their sum AC is given.

Because AB is given, a magnitude equal to it may be found; let this be DE. and because BC is given, one equal to it may be found; let this be EF. wherefore because AB is equal to DE, and BC equal to EF; the whole AC is equal to the whole DF. AC is therefore given, because DF has been found, which is equal to it.

PROP. IV.

4. IF a given magnitude be taken from a given magnitude; the remaining magnitude shall be given.

From the given magnitude AB let the given magnitude AC be taken; the remaining magnitude CB is given.

Because AB is given, a magnitude equal to it may be found; let this be DE. and because AC is given, one equal to it may be found; let this be DF. wherefore because AB is equal to DE, and AC to DF; the remainder CB is equal to the remainder FE. CB is therefore given, because FE which is equal to it has been found.

12. PROP. V.

IF of three magnitudes, the first together with the second be given, and also the second together with the third; either the first is equal to the third, or one of them is greater than the other by a given magnitude.

Let AB, BC, CD be three magnitudes, of which AB together with BC, that is AC, is given; and also BC together with CD, that is BD, is given. either AB is equal to CD, or one of them is greater than the other by a given magnitude.

Because AC, BD are each of them given, they are either equal to one another, or not equal. first, let them be equal, and because AC is equal to BD, take away the common part BC; therefore the remainder AB is equal to the remainder CD.

But if they be unequal, let AC be greater than BD, and make CE equal to BD. therefore CE is given, because BD is given, and the whole AC is given, therefore AE the remainder is given. and because EC is equal to BD, by taking BC from both, the remainder EB is equal to the remainder CD. and AE is given, wherefore AB exceeds EB, that is CD by the given magnitude AE.

PROP. VI

5. IF a magnitude has a given ratio to a part of it; it shall also have a given ratio to the remaining part of it.

Let the magnitude AB have a given ratio to AC a part of it; it has also a given ratio to the remainder BC.

Because the ratio of AB to AC is given, a ratio may be found which is the same to it. let this be the ratio of DE a given magnitude to the given magnitude DF. and because DE, DF are given, the remainder FE is given, and because AB is to AC, as DE to DF, by conversion AB is to BC, as DE to EF. therefore the ratio of AB to BC is given, because the ratio of the given magnitudes DE, EF which is the same with it has been found.

COR. From this it follows, that the parts AC, CB have a given ratio to one another. because as AB to BC, so is DE to EF; by division, AC is to CB, as DF to FE; and DF, FE are given; therefore the ratio of AC to CB is given.

PROP. VII.

6. See. N

IF two magnitudes which have a given ratio to one another, be added together; the whole magnitude shall have to each of them a given ratio.

Let the magnitudes AB, BC which have a given ratio to one another, be added together; the whole AC has to each of the magnitudes AB, BC a given ratio.

Because the ratio of AB to BC is given, a ratio maybe found which is the same with it; let this be the ratio of the given magnitudes DE, EF. and because DE, DF are given, the whole DF is given. and because as AB to BC, so is DE to EF; by composition, AC is to CB, as DF to FE; and by conversion, AC is to AB, as DF to DE. wherefore because AC is to each of the magnitudes AB, BC, as DF to each of the others DE, EF; the ratio of AC to each of the magnitudes AB, BC is given.

PROP. VIII.

7. IF a given magnitude be divided into two parts which have a given ratio to one another, and if a fourth proportional can be found to the sum of the two magnitudes by which the given ratio is exhibited, one of them, and the given magnitude; each of the parts is given.

Let the given magnitude AB be divided into the parts AC, CB which have a given ratio to one another; if a fourth proportional can be found to the above-named magnitudes; AC and CB are each of them given. Because the ratio of AC to CB is given, the ratio of AB to BC is given; therefore a ratio which is:

the same with it can be found, let this be the ratio of the given magnitudes DE, EF. and because the given magnitude AB has to BC the given ratio of DE to EF, if unto DE, EF, AB a fourth proportional can be found, this which is BC is given; and because AB is given the other part AC is given.

In the same manner, and with the like limitation, If the difference AC of two magnitudes AB, BC which have a given ratio be given; each of the magnitudes

AB, BC is given.

PROP. IX.

8. MAGNITUDES which have given ratios to the same magnitude, have also a given ratio to one another.

Let A, C have each of them a given ratio to B; A has a given ratio to C.

Because the ratio of A to B is given, a ratio which is the same to it may be found; let this be the ratio of the given magnitudes D, E. and because the ratio of B to C is given, a ratio which is the same with it may be found; let this be the ratio of the given magnitudes F, G. to F, G, E find a fourth proportional H, if it can be done; and because as A is to B, so is D to E; and as B to C, so is (F to G, and so is) E to H; ex aequali, as A to C, so is D to H. therefore the ratio of A to C is given, because the ratio of the given magnitudes D and H, which is the same with it, has been found. but if a fourth proportional to F, G, E cannot be found, then it can only be said that the ratio of A to C is compounded of the ratios of A to B, and B to C, that is of the given ratios of D to E, and F to G.

PROP. X.

9. IF two or more magnitudes have given ratios to one another, and if they have given ratios, tho' they be not the same, to some other magnitudes; these other magnitudes shall also have given ratios to one another.

Let two or more magnitudes A, B, C have given ratios to one another; and let them have given ratios, tho' they be not the same, to some other magnitudes D, E, F. the magnitudes D, E, F have given ratios to one another.

Because the ratio of A to B is given, and likewise the ratio of A to D; therefore the ratio of D to B is given; but the ratio of B to E is given, therefore the ratio of D to E is given, and because the ratio of B to C is given, and also the ratio of B to E; the ratio of E to C is given. and the ratio of C to F is given; wherefore the ratio of E to F is given. D, E, F have therefore given ratios to one another.

PROP. XI.

22. IF two magnitudes have each of them a given ratio to another magnitude; both of them together shall have a given ratio to that other.

Let the magnitudes AB, BC have a given ratio to the magnitude D; AC has a given ratio to the same D.

Because AB, BC have each of them a given ratio to D, the ratio of AB to BC is given. and by composition, the ratio of AC to CB is given. but the ratio of BC to D is given; therefore the ratio of AC to D is given.

PROP. XII.

23. IF the whole have to the whole a given ratio, and the parts have to the parts given, but not the same, ratios, every one of them, whole or part, shall have to every one a given ratio.

Let the whole AB have a given ratio to the whole CD, and the parts AE, EB have given, but not the same, ratios to the parts CF, FD; every one shall have to every one, whole or part, a given ratio.

Because the ratio of AE to CF is given, as AE to CF, so make AB to CG; the ratio therefore of AB to CG is given; wherefore the ratio of the remainder EB to the remainder FG is given, because it is the same with the ratio of AB to CG. and the ratio of EB to FD is given, wherefore the ratio of FD to FG is given; and by conversion, the ratio of FD to DG is given. and because AB has to each of the magnitudes CD, CG a given ratio, the ratio of CD to CG is given; and therefore the ratio of CD to DG is given. but the ratio of GD to DF is given, wherefore the ratio of CD to DF is given, and consequently the ratio of CF to FD is given; but the ratio of CF to AE is given, as also the ratio of FD to EB; wherefore the ratio of AE to EB is given; as also the ratio of AB to each of them. the ratio therefore of every one to every one is given.

PROP. XIII.

24. IF the first of three proportional straight lines has a given ratio to the third, the first shall also have a given ratio to the second.

Let A, B, C be three proportional straight lines, that is as A to B, so is B to C; if A has to C a given ratio, A shall also have to B a given ratio.

Because the ratio of A to C is given, a ratio which is the same with it may be found; let this be the ratio of the given straight lines D, E; and between D and E find a mean proportional F;

therefore the rectangle contained by D and E is equal to the square of F, and the rectangle D, E is given because its sides D, E are given; wherefore the square of F, and the straight line F is given. and because as A is to C, so is D to

E; but as A to C, so is the square of A to the square of B; and as D to E, so is the square of D to the square of F; therefore the square of A is to the square of B, as the square of D to the square of F. as therefore the straight line A to the straight line B, so is the straight line D to the straight line F. therefore the ratio of A to B is given, because the ratio of the given straight lines D, F which is the same with it has been found.

PROP. XIV.

A.

IF a magnitude together with a given magnitude has a given ratio to another magnitude; the excess of this other magnitude above a given magnitude has a given ratio to the first magnitude. and if the excess of a magnitude above a given magnitude has a given ratio to another magnitude; this other magnitude together with a given magnitude has a given ratio to the first magnitude.

Let the magnitude AB together with the given magnitude BE, that is AE, have a given ratio to the magnitude CD; the excess of CD above a given magnitude has a given ratio to AB.

Because the ratio of AE to CD is given, as AE to CD, so make BE to FD; therefore the ratio of BE to FD is given, and BE is given, wherefore FD is given. and because as AE to CD, so is BE to FD, the remainder AB is to the remainder CF, as AE to CD. but the ratio of AE to CD is given, therefore the ratio of AB to CF is given; that is, CF the excess of CD above the given magnitude FD has a given ratio to AB.

Next, Let the excess of the magnitude AB above the given magnitude BE, that is, let AE have a given ratio to the magnitude CD; CD together with a given magnitude has a given ratio to AB.

Because the ratio of AE to CD is given, as AE to CD, so make BE to FD; therefore the ratio of BE to FD is given, and BE is given, wherefore FD is given. and because as AE to CD, so is BE to FD, AB is to CF, as AE to CD. but the ratio of AE to CD is given, therefore the ratio of AB to CF is given; that is CF which is equal to CD together with the given magnitude DF has a given ratio to AB.

PROP. XV.

B.

IF a magnitude together with that to which another magnitude has a given ratio, be given; this other is given together with that to which the first magnitude has a given ratio. Let AB, CD be two magnitudes of which AB together with BE to which CD has a given ratio, is given; CD is given together with that magnitude to which AB has a given ratio.

Because the ratio of CD to BE is given, as BE to CD, so make AE to FD; therefore the ratio of AE to FD is given, and AE is given; wherefore FD is given. and because as BE to CD, so is AE to FD; AB is to FC, as BE to CD. and the ratio of BE to CD is given, wherefore the ratio of AB to FC is given. and FD is given, that is CD together with FC to which AB has a given ratio is given.

PROP. XVI.

10. IF the excess of a magnitude above a given magnitude, has a given ratio to another magnitude; the excess of both together above a given magnitude shall have to that other a given ratio. and if the excess of two magnitudes together above a given magnitude. has to one of them a given ratio; either the excess of the other above a given magnitude has to that one a given ratio; or the other is given together with the magnitude to which that one has a given ratio.

Let the excess of the magnitude AB above a given magnitude, have a given ratio to the magnitude BC; the excess of AC, both of them together, above a given magnitude, has a given ratio to BC.

Let AD be the given magnitude the excess of AB above which, viz. DB, has a given ratio to BC. and because DB has a given ratio to BC, the ratio of DC to CB is given, and AD is given; therefore DC, the excess of AC above the given magnitude AD, has a given ratio to BC.

Next, let the excess of two magnitudes AB, BC together above a given magnitude have to one of them BC a given ratio; either the excess of the other of them AB above a given magnitude shall have to BC a given ratio; or AB is given together with the magnitude to which BC has a given ratio.

Let AD be the given magnitude, and first let it be less than AB; and because DC the excess of AC above AD has a given ratio to BC, DB has a given ratio to BC; that is DB, the excess of AB above the given magnitude AD, has a given ratio to BC.

But let the given magnitude be greater than AB, and make AE equal to it; and because EC, the excess of AC above AE, has to BC a given ratio, BC has a given ratio to BE; and because AE is given, AB together with BE to which BC has a given ratio, is given.

PROP. XVII.

11. See. N

IF the excess of a magnitude above a given magnitude has a given ratio to another magnitude; the excess of the same first magnitude above a given magnitude, shall have a given ratio to both the magnitudes together. and if the excess of either of two magnitudes above a given magnitude has a given ratio to both magnitudes together; the excess of the same above a given magnitude shall have a given ratio to the other.

Let the excess of the magnitude AB above a given magnitude have a given ratio to the magnitude BC; the excess of AB above given magnitude has a given ratio to AC.

Let AD be the given magnitude; and because DB, the excess of AB above AD, has a given ratio to BC; the ratio of DC to DB a given. make the ratio of AD to DE the same with this ratio; therefore the ratio of AD to DE is given. and AD is given, wherefore DE, and the remainder AE are given. and because as DC to DB, so is AD to DE, AC is to EB, as DC to DB; and the ratio of DC to DB is given, wherefore the ratio of AC to EB is given. and because the ratio of EB to AC is given, and that AE is given, therefore EB the excess of AB above the given magnitude AE, has a given ratio to AC.

Next, let the excess of AB above a given magnitude have a given ratio to AB and BC together, that is to AC; the excess of AB above a given magnitude has a given ratio to BC.

Let AE be the given magnitude; and because EB the excess of AB above AE has to AC a given ratio, as AC to EB, so make AD to DE; therefore the ratio of AD to DE is given, as also the ratio of AD to AE. and AE is given, wherefore AD is given. and because as the whole, AC, to the whole, EB, so is AD to DE; the remainder DC is to the remainder DB, as AC to and the ratio of AC to EB is given, wherefore the ratio of DC to DB is given, as also the ratio of DB to BC. and AD is given, therefore DB, the excess of AB above the given magnitude AD, has a given ratio to BC.

PROP. XVIII.

14. IF to each of two magnitudes, which have a given ratio to one another, a given magnitude be added; the wholes shall either have a given ratio to one another, or the excess of one of them above a given magnitude shall have a given ratio to the other.

Let the two magnitudes AB, CD have a given ratio to one another, and to AB let the given magnitude BE be added, and the given magnitude DF to CD. the wholes AE, CF either have a given ratio to one another, or the excess of one of them above a given magnitude has a given ratio to the other.

Because BE, DF are each of them given, their ratio is given.

and if this ratio be the same with the ratio of AB to CD, the ratio of AE to CF, which is the same with the given ratio of AB to CD, shall be given.

But if the ratio of BE to DF be not the same with the ratio of AB to CD; either it is greater than the ratio of AB to CD, or, by inversion, the ratio of DF to BE is greater than the ratio of CD to AB. first, let the ratio of BE to DF be greater than the ratio of AB to CD; and as AB to CD, so make BG to DF; therefore the ratio of BG to DF is given; and DF is given, therefore BG is given. and because BE has a greater ratio to DF than (AB to CD, that is than) BG to DF, BE is greater than BG. and because as AB to CD, so is BG to DF, therefore AG is to CF, as AB to CD. but the ratio of AB to CD is given, wherefore the ratio of AG to CF is given; and because BE, BG are each of them given, GE is given. therefore AG, the excess of AE above the given magnitude GE has a given ratio to CF. the other case is demonstrated in the same manner.

PROP. XIX.

15. IF from each of two magnitudes, which have a given ratio to one another, a given magnitude be taken; the remainders shall either have a given ratio to one another, or the excess of one of them above a given magnitude, shall have a given ratio to the other.

Let the magnitudes AB, CD have a given ratio to one another, and from AB let the given magnitude AE be taken, and from CD, the given magnitude CF. the remainders EB, FD shall either have a given ratio to one another, or the excess of one of them above a given magnitude shall have a given ratio to the other.

Because AE, CF are each of them given, their ratio is given; and if this ratio be the same with the ratio of AB to CD, the ratio of the remainder EB to the remainder FD, which is the same with the given ratio of AB to CD, shall be given.

But if the ratio of AB to CD be not the same with the ratio of AE to CF, either it is greater than the ratio of AE to CF, or, by inversion, the ratio of CD to AB is greater than the ratio of CF to AE. first, let the ratio of AB to CD be greater than the radio of AE to CF, and as AB to CD, so make AG to CF; therefore the ratio of AG to CF is given, and CF is given, wherefore AG is given. and because the ratio of AB to CD, that is the ratio of AG to CF, is greater than the ratio of AE to CF; AG is greater than AE. and AG, AE are given, therefore the remainder EG is given. and as AB to CD, so is AG to CF, and so is the remainder GB to the remainder FD; and the ratio of AB to CD is given, wherefore the ratio of GB to FD is therefore GB, the excess of EB above the given magnitude DG, has a given ratio to FD. in the same manner the other case is demonstrated.

PROP. XX.

16. IF to one of two magnitudes which have a given ratio to one another, a given magnitude be added, and from the other a given magnitude be taken; the excess of the sum above a given magnitude shall have a given ratio to the remainder.

Let the two magnitudes AB, CD have a given ratio to one another, and to AB let the given magnitude EA be added, and from CD let the given magnitude CF be taken; the excess of the sum EB above a given magnitude has a given ratio to the remainder FD.

Because the ratio of AB to CD is given, make as AB to CD, so AG to CF. therefore the ratio of AG to CF is given, and CF is given, wherefore AG is given; and EA is given, therefore the whole EG is given. and because as AB to CD, so is AG to CF, and so is the remainder GB to the remainder FD; the ratio of GB to FD is given. and EG is given,

therefore GB, the excess of the sum EB above the given magnitude EG, has a given ratio to the remainder FD.

PROP. XXI.

C.

IF two magnitudes have a given ratio to one another, if a given magnitude be added to one of them, and the other be taken from a given magnitude; the sum together with the magnitude to which the remainder has a given ratio, is given. and the remainder is given together with the magnitude to which the sum has a given ratio.

Let the two magnitudes AB, CD have a given ratio to one another; and to AB let the given magnitude BE be added, and let CD be taken from the given magnitude FD. the sum AE is given together with the magnitude to which the remainder FC has a given ratio.

Because the ratio of AB to CD is given, make as AB to CD, so GB to FD. therefore the ratio of GB to FD is given, and FD is given, wherefore GB is given; and BE is given, the whole GE is therefore given. and because as AB to CD, so is GB to FD, and so is GA to FC; the ratio of GA to FC is given. and AE together with GA is given, because GE is given; therefore the sum AE together with GA to which the remainder FC has a given ratio, is given. the second part is manifest from Pro.

PROP. XXII.

See. N

IF two magnitudes have a given ratio to one another, if from one of them a given magnitude be taken, and the other be taken from a given magnitude; each of the remainders is given together with the magnitude to which the other remainder has a given ratio.

Let the two magnitudes AB, CD have a given ratio to one another, and from AB let the given magnitude AE be taken, and let CD be taken from the given magnitude CF; the remainder EB is given together with the magnitude to which the other remainder DF has a given ratio.

Because the ratio of AB to CD is given, make as AB to CD, so AG to CF. the ratio of AG to CF is therefore given, and CF is given, wherefore AG is given; and AE is given, and therefore the remainder EG is given. and because as AB to CD, so is AG to CE, and so is the remainder BG to the remainder DF; the ratio of BG to DF is given. and EB together with BG is given, because EG is given. therefore the remainder EB together with BG to which DF the other remainder has a given ratio is given. the second part is plain from Pro.

PROP. XXIII.

20. IF from two given magnitudes there be taken magnitudes which have a given radio to one another, the remainders shall either have a given ratio to one another, or the excess of one of them above a given magnitude shall have a given ratio to the other.

Let AB, CD be two given magnitudes, and from them let the magnitudes AE, CF which have a given ratio to one another be taken; the remainders EB, FD either have a given ratio to one another, or the excess or one or them above a given magnitude has a given ratio to the other.

Because AB, CD are each of them given, the ratio of AB to CD is given. and if this ratio be the same with the ratio of AE to CF, then the remainder EB has the same given ratio to the remainder FD.

But if the ratio of AB to CD be not the same with the ratio of AE to CF, it is either greater than it, or, by inversion, the ratio of CD to AB is greater than the ratio CF to AE. first, let the ratio of AB to CD be greater than the ratio of AE to CF; and as AE to CF, so make AG to CD, therefore the ratio of AG to CD is given, because the ratio of AE to CF is given; and CD is given, wherefore AG is given; and because the ratio of AB to CD is greater than the ratio of (AE to CF, that is, than the ratio of) AG to CD; AB is greater than AG. and AB, AG are given, therefore the remainder BG is given. and because as AE to CF, so is AG to CD, and so is EG to FD; the ratio of EG to FD is given. and GB is given, therefore EG the excess of EB above the given magnitude GB, has a given ratio to FD. the other case is shewn in the same way.

PROP. XXIV.

13. IF there be three magnitudes, the first of which has a given ratio to the second, and the excess of the second above a given magnitude has a given ratio to the third; the excess of the first above a given magnitude shall also have a given ratio to the third.

Let AB, CD, E be three magnitudes, of which AB has a given ratio to CD; and the excess of CD above a given magnitude has a given ratio to E. the excess of AB above a given magnitude has a given ratio to E.

Let CF be the given magnitude the excess of CD above which, viz. FD has a given ratio to E. and because the ratio of AB to CD is given, as AB to CD so make AG to CF; therefore the ratio of AG to CF is given; and CF is given, wherefore AG is given. and because as AB to CD, so is AG to CF, and so is. GB to FD; the ratio of GB to FD is given. and the ratio of FD to E is given, wherefore the ratio of GB to E is given. and AG is given, therefore GB the excess of AB above the given magnitude AG has a given ratio to E.

COR. 1. And if the first has a given ratio to the second, and the excess of the first above a given magnitude has a given ratio to the third; the excess of the second above a given magnitude shall have a given ratio to the third. for if the second be called the first, and the first the second, this Corollary will be the same with the Proposition.

COR. 2. Also if the first has a given ratio to the second, the excess of the third above a given magnitude has also a given ratio to the second, the same excess shall have a given ratio to the first; as is evident from the 9th Dat.

PROP. XXV.

17. IF there be three magnitudes, the excess of the first where of above a given magnitude has a given ratio to the second; and the excess of the third above a given magnitude has a given ratio to the same second. the first shall either have a given ratio to the third, or the excess of one of them above a given magnitude shall have a given ratio to the other.

Let AB, C, DE, be three magnitudes, and let the excesses of each of the two AB, DE above given magnitudes have given ratios to C; AB, DE either have a given ratio to one another, or the excess of one of them above a given magnitude has a given ratio to the other.

Let FB the excess of AB above the given magnitude AF have a given ratio to C; and let GE the excess of DE above the given magnitude DG have a given ratio to C; and because FB, GE have each of them a given ratio to C, they have a given ratio to one another. but to FB, GE the given magnitudes AF, DG are added; therefore the whole magnitudes AB, DE have either a given ratio to one another, or the excess of one of them above a given magnitude has a given ratio to the other.

PROP. XXVI.

18. IF there be three magnitudes the excesses of one of which above given magnitudes have given ratios to the other two magnitudes; these two shall either have a given ratio to one another, or the excess of one of them above a given magnitude shall have a given ratio to the other.

Let AB, CD, EF be three magnitudes, and let GD the excess of one of them CD above the given magnitude CG have a given ratio to AB; and also let KD the excess of the same CD above the given magnitude CK have a given ratio to EF. either AB has a given ratio to EF, or the excess of one of them above a given magnitude has a given ratio to the other.

Because GD has a given ratio to AB, as GD to AB, so make CG to HA; therefore the ratio of CG to HA is given; and CG is given, wherefore HA is given. and because as GD to AB, so is CG to HA, and so is CD to HB; the ratio of CD to HB is given. also because KD has a given ratio to EF, as KD to EF, so make CK to LE; therefore the ratio of CK to LE is given; and CK is given, wherefore LE is given. and because as KD to EF, so is CK to LE, and so is CD to LF; the ratio of CD to LF is given. but the ratio of CD to HB is given, wherefore the ratio of HB to LF is given. and from HB, LF the given magnitudes HA, LE being taken, the remainders AB, EF shall either have a given ratio to one another, or the excess of one of them above a given magnitude has a given ratio to the other.

“Another Demonstration.

Let AB, C, DE be three magnitudes, and let the excesses of one of them C above given magnitudes have given ratios to AB and DE. either AB, DE have a given ratio to one another, or the excess of one of them above a given magnitude has a given ratio to the other.

Because the excess of C above a given magnitude has a given ratio to AB, therefore AB together with a given magnitude has a given ratio to C. let this given magnitude be AF, wherefore FB has a given ratio to C. also, because the excess of C above a given magnitude has a given ratio to DE, therefore DE together with a given magnitude has a given ratio to C. let this given magnitude be DG, wherefore GE has a given ratio to C. and FB has a given ratio to C, therefore the ratio of FB to GE is given. and from FB, GE the given magnitudes AF, DG being taken, the remainders AB, DE either have a given ratio to one another, or the excess of one of them above a given magnitude has a given ratio to the other."

PROP. XXVII.

19. IF there be three magnitudes the excess of the first of which above a given magnitude has a given ratio to the second; and the excess of the second above a given magnitude has also a given ratio to the third. the excess of the first above a given magnitude shall have a given ratio to the third.

Let AB, CD, E be three magnitudes the excess of the first of which AB above the given magnitude AG, viz. GB has a given ratio to CD; and FD the excess of CD above the given magnitude CF, has a given ratio to E. the excess of AB above a given magnitude has a given ratio to E.

Because the ratio of GB to CD is given, as GB to CD, so make GH to CF; therefore the ratio of GH to CF is given; and CF is given, wherefore GH is given; and AG is given, wherefore the whole AH is given. and because as GB to CD, so is GH to CF, and so is the remainder HB to the remainder FB; the ratio of HB to FD is given. and the ratio of FD to E is given. wherefore the ratio of HB to E is given. and AH is given; therefore HB the excess of AB above the given magnitude AH has a given ratio to E.

"Otherwise.

Let AB, C, D be three magnitudes, the excess EB of the first of which AB above the given magnitude AE has a given ratio to C, and the excess of C above a given magnitude has a given ratio to D. the excess of AB above a given magnitude has a given ratio to D.

Because EB has a given ratio to C, and the excess of C above a given magnitude has a given ratio to D; therefore the excess of EB above a given magnitude has a given ratio to D. let this given magnitude be EF, therefore FB the excess of EB above EF has a given ratio to D. and AF is given, because AE, EF are given.

therefore FB the excess of AB above the given magnitude AF has a given ratio to D."

PROP. XXVIII.

25. IF two lines given in position cut one another, the point or points in which they cut one another are given.

Let two lines AB, CD given in position cut one another in the point E; the point E is given.

Because the lines AB, CD are given in position, they have always the same situation, and therefore the point, or points, in which they cut one another have always the same situation. and because the lines AB, CD can be found, the point, or points, in which they cut one another, are likewise found; and therefore are given in position.

PROP. XXIX.

26. IF the extremities of a straight line be given in position; the straight line is given in position and magnitude.

Because the extremities of the straight line are given, they can be found; let these be the points A, B, between which a straight line AB can be drawn; this has an invariable position, because between two given points there can be drawn but one straight line. and when the straight line AB is drawn, its magnitude is at the same time exhibited, or given. therefore the straight line AB is given in position and magnitude.

PROP. XXX.

27. IF one of the extremities of a straight line given in position and magnitude be given; the other extremity shall also be given.

Let the point A be given, to wit one of the extremities of straight line given in magnitude, and which lies in the straight line AC given in position; the other extremity is also given.

Because the straight line is given in magnitude, one equal to can be found; let this be the straight line D. from the grace straight line AC cut off AB equal to the lesser D. therefore the other extremity B of the straight line AB is found. and the point B has always the same situation, because any other point in AC, upon the same side of A, cuts off between it and the point A greater or less straight line than AB, that is than D. therefore the point B is given. and it is plain another such point can be found in AC produced upon the other side of the point A.

PROP. XXXI.

28. IF a straight line be drawn through a given point parallel to a straight line given in position; that straight line is given in position.

Let A be a given point, and BC a straight line given in position; the straight line drawn thro' A parallel to BC is given in position.

Thro' A draw the straight line DAE parallel to BC; the straight line DAE has always the same position, because no other straight line can be drawn through A parallel to BC. therefore the straight line DAE which has been found is given in position,

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PROP. XXXII.

IF a straight line be drawn to a given point in a given straight line, and makes a given angle with it. that straight line is given in position.

Let AB be a straight line given in position, and C a given point in it, the straight line drawn to C which makes a given angle with CB, is given in position.

Because the angle is given, one equal to it can be found; let this be the angle at D. at the given point C in the given straight line AB make the angle ECB equal to the angle at D. therefore the straight line EC has always the same situation, because any other straight line FC drawn to the point C makes with CB a greater or less angle than the angle ECB or the angle at D. therefore the straight line EC which has been found is given in position.

It is to be observed that there are two straight lines EC, GC upon one side of AB that make equal angles with it, and which make equal angles with it when produced to the other side.

PROP. XXXIII.

IF a straight line be drawn from a given point, to a straight line given in position, and makes a given angle with it; that straight line is given in position.

From the given point A let the straight line AD be drawn to the straight line BC given in position, and make with it a given angle ADC; AD is given in position.

Thro' the point A draw the straight line EAF parallel to BC; and because thro' the given point A the straight line EAF is drawn parallel to BC which is given in position, EAF is therefore given in position. and because the straight line AD meets the parallels BC, EF, the angle EAD is equal to the angle ADC; and

ADC is:

given, wherefore also the angle EAD is given. therefore because the straight line DA is drawn to the given point A in the straight line EF given in position, and makes with it a given angle EAD; AD is given in position.

PROP. XXXIV.

31. IF from a given point to a straight line given in position, a straight line be drawn which is given in magnitude; the same is also given in position.

Let A be a given point, and BC a straight line given in position; a straight line given in magnitude drawn from the position A to BC is given in position.

Because the straight line is given in magnitude, one equal to it can be found; let this be the straight line D. from the point A draw AE perpendicular to BC; and because AE is the shortest of all the straight lines which can be drawn from the point A to BC, the straight line D, to which one equal is to be drawn from the point A to BC, cannot be less than AE. If therefore D be equal to AE, AE is the straight line given in magnitude drawn from the point A to BC. and it is evident that AE is given in position because it is drawn from the given point A to BC which is given position, and makes with BC the given angle AEC.

But if the straight line D be not equal to AE, it must be greater than it. produce AE, and make AF equal to D; and from the center A, at the distance AF describe the circle GFH, and join AC, AH. because the circle GFH is given in position, and the straight line BC is also given in position; therefore their intersection G is given; and the point A is given; wherefore AG is given in position, that is, the straight line AG given in magnitude (for it is equal to D) and drawn from the given point A to the straight line BC given in position, is also given in position. and in like manner AH is given in position. therefore, in this case there are two straight lines AG,

AH of the same given magnitude which can be drawn from a given point A to a straight line BC given in position.

PROP. XXXV.

32. IF a straight line be drawn between two parallel straight lines given in position, and makes given angles with them; the straight line is given in magnitude.

Let the straight line EF be drawn between the parallels AB, CD which are given in position, and make the given angles BEF, EFD; EF is given in magnitude.

In CD take the given point G, and thro' G draw GH parallel to EF. and because CD meets the parallels GH, EF, the angle EFD is equal to the angle HGD. and EFD is a given angle, wherefore the angle HGD is given and because HG is drawn to the given point G in the straight line CD given in position, and makes a given angle HGD; the straight line HG is given in position and AB is given in position, therefore the point H is given; and the point G is also given, wherefore GH is given in magnitude and EF is equal to it; therefore EF is given in magnitude.

PROP. XXXVI.

33. IF a straight line given in magnitude be drawn between two parallel straight lines given in position; it shall make given angles with the parallels.

Let the straight line EF given in magnitude be drawn between the parallel straight lines AB, CD which are given in position; the angles AEF, EFC shall be given.

Because EF is given in magnitude, a straight line equal to it can be found; let this be G. in AB take a given point H, and from it draw HK perpendicular to CD. therefore the straight line G, that is EF, cannot be less than HK.

and if G be equal to HK, EF also is equal to it; wherefore EF is at right angles to CD, for if it be not, EF would be greater than HK, which is absurd. therefore the angle EFD is a right and consequently a given angle.

But if the straight line G be not equal to HK, it must be greater than it. produce HK, and take HL equal to G; and from the center H, at the distance HL describe the circle MLN, and join HM, HN. and because the circle MLN, and the straight line CD are given in position, the points M, N are given; and the point H is given, wherefore the straight lines HM, HN are given in position. and CD is given in position, therefore the angles HMN, HNM are given in position. of the straight lines HM, HN let HN be that which is not parallel to EF, for EF cannot be parallel to both of them; and draw EO parallel to HN. EO therefore is equal to HN, that is to G; and EF is equal to G, wherefore EO is equal to EF, and the angle EFO to the angle EOF, that is to the given angle HNM. and because the angle HNM which is equal to the angle EFO or EFD has been found, therefore the angle EFD, that is the angle AEF, is given in magnitude, and consequently the angle EFC.

PROP. XXXVII.

E.

IF a straight line given in magnitude be drawn from a point to a straight line given in position, in a given angle; the straight line drawn thro' that point parallel to the straight line given in position, is given in position.

Let the straight line AD given in magnitude be drawn from the point A to the straight line BC given in position, in the given angle ADC; the straight line EAF drawn through A parallel to BC is given in position.

In BC take a given point G, and draw GH parallel to AD. and because HG is drawn to a given point G in the straight line BC given in position, in a given:

angle HGC, for it is equal to the given angle ADC; HG is given in position; but it is given also in magnitude, because it is equal to AD which is given in magnitude, therefore because G one of the extremities of the straight line GH given in position and magnitude is given, the other extremity H is given. and the straight line EAF which is drawn through the given point H parallel to BC given in position, is therefore given in position.

PROP. XXXVIII.

34. IF a straight line be drawn from a given point to two parallel straight lines given in position; the ratio of the segments between the given point and the parallels shall be given.

Let the straight line EFG be drawn from the given point E to the parallels AB, CD; the ratio of EF to EG is given.

From the point E draw EHK perpendicular to CD. and because from a given point E the straight line EK is drawn to CD which is given in position, in a given angle EKC; EK is given in position.

and AB, CD are given in position; therefore the points are H, K are given. and the point E is given, wherefore EH, EK are given in magnitude, and the ratio of them is therefore given. but as EH to EK, so is EF to EG, because AB, CD are parallels. therefore the ratio of EF to EG is given.

PROP. XXXIX.

35. 36.

IF the ratio of the segments of a straight line between a given point in it and two parallel straight lines be given; if one of the parallels be given in position, the other is also given in position.

From the given point A let the straight line AED be drawn to the two parallel straight lines FG, BC, and let the ratio of the segments AE, AD be given; if one of the parallels BC be given in position, the other FG is also given in position.

From the point A draw AH perpendicular to BC, and let it meet FG in K. and because AH is drawn from the given point A to the straight line BC given in position, and makes a given angle AHD; AH is given in position. and BC is likewise given in position, therefore the point H is given. the point A is also given, wherefore AH is given in magnitude. and, because FG, BC are parallels, as AE to AD, so is AK to AH; and the ratio of AE to AD is given, wherefore the ratio of AK to AH is given; but AH is given in magnitude, therefore AK is given in magnitude; and it is also given in position, and the point A is given; wherefore the point K is given. and because the straight line FG is drawn thro' the given point K parallel to BC which is given in position, therefore FG is given in position.

PROP. XL.

37. 38.

IF the ratio of the segments of a straight line into which it is cut by three parallel straight lines, be given; If two of the parallels are given in position, the third also is given in position.

Let AB, CD, HK be three parallel straight lines, of which AB, CD are given in position; and let the ratio of the segments GE,

GF into which the straight line GEF is cut by the three parallels, be given; the third parallel HK is given in position.

In AB take a given point L, and draw LM perpendicular to CD, meeting HK in N. because LM is drawn from the given point L to CD which is given in position, and makes a given angle LMD; LM is given in position. and CD is given in position, wherefore the point M is given; and the point L is given, LM is therefore given in magnitude. and because the ratio of GS, to

GF is given, and as GE to GF, so is NL to NM; the ratio of NL to NM is given; and therefore the ratio of ML to LN is given. but LM is given in magnitude, wherefore LN is given magnitude; and it is also given in position, and the point L is given; wherefore the point N is given, and because the straight line HK is drawn thro' the given point N parallel to CD which is given in position, therefore HK is given in position.

PROP. XLI.

F.

IF a straight line meets three parallel straight lines which are given in position; the segments into which they cut it, have a given ratio.

Let the parallel straight lines AB, CD, EF given in position be cut by the straight line GHK; the ratio of GH to HK is given.

In AB take a given point L, and draw LM perpendicular to CD, meeting EF in N; therefore LM is given in position; and CD, EF are given in position, wherefore the points M, N are given. and the point L is given, therefore the straight lines LM, MN are given in magnitude; and the ratio of LM to MN is therefore given, but as LM to MN, so is GH to HK; wherefore the ratio of GH to HK is given.

PROP. XLII.

39. See. N.

IF each of the sides of a triangle be given in magnitude; the triangle is given in species.

Let each of the sides of the triangle ABC be given in magnitude; the triangle ABC is given in species.

Make a triangle DEF the sides of which are equal, each to each, to the given straight lines AB, BC, CA; which can be done, because any two of them must be greater than the third; and let DE be equal to AB, EF to BC, and FD to CA. and because the two sides ED, DF are equal to the two BA, AC, each to each, and the base EF equal to the base BC; the angle EDF is equal to the angle BAC. therefore because the angle EDF, which is equal to the angle BAC, has been found, the angle BAC is given, in like manner the angles at B, C are given, and because the sides AB, BC, CA are given, their ratios to one another are given. therefore the triangle ABC is given in species.

PROP. XLIII.

IF each of the angles of a triangle be given in magnitude; the triangle is given in species.

Let each of the angles of the triangle ABC be given in magnitude; the triangle ABC is given in species.

Take a straight line DE given in position and magnitude, and at the points D, E make the angle EDF equal to the angle BAC, and the angle DEF equal to ABC; therefore the other angles EFD, BCA are equal. and each of the angles at the points A, B, C is given, wherefore

each of those at the points D, E, F is given. and because the straight line FD is drawn to the given point D in DE which is given in position, making the given angle EDF; therefore DF is given in position in like manner EF also is given in position; wherefore the point F is given, and the points D, E are given; therefore each of the straight lines DE, EF, FD is given in magnitude, wherefore the triangle DEF is given in species; and it is similar to the triangle ABC; which therefore is given in species.

PROP. XLIV.

41. IF one of the angles of a triangle be given, and if the sides about it have a given ratio to one another; the triangle is given in species.

Let the triangle ABC have one of its angles BAC given, and let the sides BA, AC about it have a given ratio to one another; the triangle ABC is given in species.

Take a straight line DE given in position and magnitude, and at the point D in the given straight line DE make the angle EDF equal to the given angle BAC; wherefore the angle EDF is given. and because the straight line FD is drawn to the given point D in ED which is given in position, making the given angle EDF; therefore FD is given in position and because the ratio of BA to AC is given, make the ratio of ED to DF the same with it, and join EF. and because the ratio of ED to DF is given, and ED is given, therefore DF is given in magnitude; and it is given also in position, and the point D is given, wherefore the point F is given. and the points D, E are given, wherefore DE, EF, FD are given in magnitude; and the triangle DEF is therefore given in species. and because the triangles ABC, DEF have one angle BAC equal to one angle EDF, and the sides about these angles proportionals; the triangles are similar. but the triangle DEF is given in species, and therefore also the triangle ABC.

PROP. XLV.

42. IF the sides of a triangle have to one another given ratios; the triangle is given in species.

Let the sides of the triangle ABC have given ratios to one another. the triangle ABC is given in species.

Take a straight line D given in magnitude; and because the ratio of AB to LC is given, make the ratio of D to E the same with it; and D is. given, therefore E is given. and because the ratio of BC to CA is given, to this make the ratio of E to F the same; and E is given, and therefore F. and because as AB to BC, so is D to E, by composition AB and BC together are to BC, as D and E to E; but as BC to CA, so is E to F; therefore, ex aequali, as AB and BC are to CA, so are D and E to F. and AB and BC are greater than CA, therefore D and E are greater then F. in the same another any two of the there D, E, F are greater the then the GHK when sides are equal to D, E, F, so that GH be equal to D, HK to E, and KG to F. and because D, E, F are, each of them, given, therefore GH, HK, KG are each of them given in magnitude; therefore the triangle GHK is given in species. but as AB to BC, so is (D to E, that is) GH to HK; and as BC to CA, so is (E to F, that is) HK to KG; therefore, ex acquali, as AB to AC, so is GH to GK. wherefore the triangle ABC is equiangular and similar to the triangle GHK. and the triangle GHK is given in species; therefore also the triangle ABC is given in species.

COR. If a triangle is required to be made the sides of which shall have the same ratios which three given straight lines D, E, F have to one another; it is necessary that every two of them be greater than the third.

PROP. XLVI.

43. IF the sides of a right angled triangle about one of the acute angles have a given ratio to one another; the triangle is given in species.

Let the sides AB, BC about the acute angle ABC of the triangle ABC which has a right angle at A, have a given ratio to one another; the triangle ABC is given in species.

Take a straight line DE given in position and magnitude; and because the ratio of AB to BC is given, make as AB to BC, so DE to EF; and because DE has a given ratio to EF, and DE is given, therefore EF is given. and because as AB to BC, so is DE to EF, and AB is less than BC, therefore DE is less than EF. from the point D draw DG at right angles to DE, and from the center E at the distance EF describe a circle which shall meet DG in two points, let G be either of them, and join EG; therefore the circumference of the circle is given in position. and the straight line DG is given in position, because it is drawn to the given point D in DE given in position, in a given angle, therefore the point G is given, and the points D, E are given, wherefore DE, EG, GD are given in magnitude, and the triangle DEG in species. and because the triangles ABC, DEG have the angle BAC equal to the angle EDG, and the sides about the angles ABC, DEG proportionals, and each of the other angles BCA, EGD less than a right angle; the triangle ABC is equiangular and similar to the triangle DEG. but DEG is given in species, therefore the triangle ABC is given in species. and in the same manner, the triangle made by drawing a straight line from E to the other point in which the circle meets DG is given in species.

PROP. XLVII.

44. IF a triangle has one of its angles which is not a right angle given, and if the sides about another angle have a given ratio to one another; the triangle is given in species.

Let the triangle ABC have one of its angles ABC a given, but not a right angle, and let the sides BA, AC about another angle BAC have a given ratio to one another; the triangle ABC is given in species.

First, Let the given ratio be the ratio of equality, that is, let the sides BA, AC and consequently the angles ABC, ACB be equal. and because the angle ABC is given, the angle ACB, and also the remaining angle BAC is given. therefore the triangle ABC is given in species. and it is evident that in this case the given angle ABC must be acute.

Next, Let the given ratio be the ratio of a less to a greater, that is, let the side AB adjacent to the given angle be less than the side AC. take a straight line DE given in position and magnitude, and make the angle DEF equal to the given angle ABC; therefore EF is given in position. and because the ratio of BA to AC is given, as BA to AC, so make ED to DG; and because the ratio of ED to

DG is given, and ED is given, the straight line DG is given. and BA is less than AC, therefore ED is less than DG. from the center D, at the distance DG describe the circle GF meeting EF in F, and join DF. and because the circle is given in position, as also the straight line EF, the point F is given. and the points D, E are given, wherefore the straight lines DE, EF, FD are given in magnitude, and the triangle DEF in species. and because BA is less than AC, the angle ACB is less than the angle ABC, and therefore ACB is less than a right angle. in the same manner, because ED is less than DG or DF, the angle DFE is less than a right angle, and because the triangles ABC, DEF have the angle ABC equal to the angle DEF, and the sides about the angles BAC, EDF proportionals, and each of the other angles ACB, DFE less than a right angle; the triangles ABC, DEF are similar. and DEF is given in species, wherefore the triangle ABC is also given in species.

Thirdly, Let the given ratio be the ratio of a greater to a less, that is, let the side AB adjacent to the given angle be greater than AC. and, as in the last case, take a straight line DE given in position and magnitude, and make the angle DEF equal to the given angle ABC; therefore EF is given in position. also draw DG perpendicular to EF; therefore if the ratio of BA to AC be the same with the ratio of ED to the perpendicular DG, the triangles ABC, DEG are similar, because the angles ABC, DEG are equal, and DGE is a right angle. therefore the angle ACB is a right angle, and the triangle ABC is given in species.

But if, in this last case, the given ratio of BA to AC be not the same with the ratio of ED to DG, that is, with the ratio of BA to the perpendicular AM drawn from A to BC; the ratio of BA to AC must be less than the ratio of BA to AM, because AC is greater than AM. make as BA to AC, so ED to DH; therefore the ratio of ED to DH is less than the ratio of (BA to AM, that is than the ratio of) ED to DG; and consequently DH is greater P than DG; and because BA is greater than AC, ED is greater than DH. from the center D, at the distance DH, describe the circle KHF which necessarily meets the straight line EF in two points, because DH is greater than DG, and less than DE. let the circle meet EF in the points F, K which are given, as was shewn in the preceding case; and, DF, DK being joined, the triangles DEF, DEK are given in species, as was there shewn. from the center A, at the distance AC describe a circle meeting BC again in L. and if the angle ACB be less than a right angle, ALB must be greater than a right angle; and on the contrary. in the same manner, if the angle DFE be less than a right angle, DKE must be greater than one; and on the contrary. let each of the angles ACB, DFE be either less or greater than a right angle; and because in the triangles ABC, DEF the angles ABC, DEF are equal, and the sides BA, AC, and ED, DF about two of the other angles proportionals, the triangle ABC is similar to the triangle DEF. in the same manner, the triangle ABL is similar to DEK. and the triangles DEF, DEK are given in species, therefore also the triangles ABC, ABL are given in species. and from this it is evident, that, in this third case, there are always two triangles of a different species to which the things mentioned as given in the Proposition can agree.

PROP. XLVIII.

45. IF a triangle has one angle given, and if both the sides together about that angle have a given ratio to the remaining side; the triangle is given in species.

Let the triangle ABC have the angle BAC given, and let the sides BA, AC together about that angle have a given ratio to BC; the triangle ABC is given in species.

Bisect the angle BAC by the straight line AD; therefore the angle BAD is given. and because as BA to AC, so is BD to DC, by permutation, as AB to BD, so is AC to CD; and as BA and AC together to BC, so is AB to BD. but the ratio of BA and AC together to BC is given, wherefore the ratio of AB to BD is given; and the angle BAD is given, therefore the triangle ABD is given in species. and the angle ABD is therefore given.; the angle BAC is also given, wherefore the triangle ABC is given in species.

A triangle which shall have the things that are mentioned in the Proposition to be given, can be found in the following manner. let EFG be the given angle, and let the ratio of H to K be the

given ratio which the two sides about the angle EFG must have to the third side of the triangle, therefore because two sides of a triangle are greater than the third side, the ratio of H to K must be the ratio of a greater to a less. bisect the angle EFG by the straight line FL, and by the 47th Proposition find a triangle of which EFL is one of the angles, and in which the ratio of the sides about the angle opposite to FL is the same with the ratio of H to K; to do which, take FE given in position and magnitude, and draw EL perpendicular to FL. then, if the ratio of H to K be the same with the ratio of FE to EL, produce EL and let it meet FG in P; the triangle FEP is that which was to be found. for it has the given angle EFG, and because this angle is bisected by FL, the sides EF, FP together are to EP, as FE to EL, that is as H to K.

But if the ratio of H to K be not the same with the ratio of FE to EL, it must be less than it, as was shewn in Pro. and in this case there are two triangles each of which has the given angle EFL, and the ratio of the sides about the angle opposite to FL the same with the ratio of H to K. by Pro. find these triangles EFM, EFN each of which has the angle EFL for one of its angles, and the ratio of the side FE to EM or EN the same with the ratio of H to K; and let the angle EMF be greater, and ENF less than a right angle. and because H is greater than K, EF is greater than EN, and therefore the angle EFN, that is the angle NFG, is less than the angle ENF. to each of these add the angles NEF, EFN; therefore the angles NEF, EFG are less than the angles NEF, EFN, FNE, that is than two right angles; therefore the straight lines EN, FG must meet together when produced; let them meet in O, and produce EM to G. each of the triangles EFG, EFO has the things mentioned to be given in the Proposition. for each of them has the given angle EFG, and because this angle is bisected by the straight line FMN, the sides EF, FG together have to EG the third side the ratio of FE to EM, that is of H to K. in like manner, the sides EF, FO together have to EO the ratio which H has to K.

PROP. XLIX.

46. IF a triangle has one angle given, and if the sides about another angle, both together, have a given ratio to the third side; the triangle is given in species.

Let the triangle ABC have one angle ABC given, and let the two sides BA, AC about another angle BAC have a given ratio to BC; the triangle ABC is given in species.

Suppose the angle BAC to be bisected by the straight line AD; BA and AC together are to BC, as AB to BD, as was shewn in the preceding Proposition. but the ratio of BA and AC together to BC is given, therefore also the ratio of AB to BD is given. and the angle ABD is given, wherefore the triangle ABD is given in species; and consequently the angle BAD, and its double the angle BAC are given; and the angle ABC is given. therefore the triangle ABC is given in species.

A triangle which shall have the things mentioned in the Proposition to be given, may be thus found. Let EFG be the given angle, and the ratio of H to K the given ratio; and by Pro. find the triangle EFL which has the angle EFG for one of its angles, and the ratio of the sides EF, FL about this angle the same with the ratio of H to K; and make the angle LEM equal to the angle FEL. and because the ratio of H to K is the ratio which two sides of a triangle have to the third, H must be greater than K; and because EF is to FL, as H to K, therefore EF is greater than FL, and the angle FEL, that is LEM is therefore less than the angle ELF. wherefore the angles LFE, FEM are less than two right angles, as was shewn in the foregoing Proposition, and the straight lines FL, EM must meet if produced; let them meet in G. EFG is the triangle which was to be found; for EFG is one of its angles, and because the angle FEG is bisected by EL, the two sides FE, EG together have to the third side FG the ratio of EF to FL, that is the given ratio of H to K.

PROP. L.

76. IF from the vertex of a triangle given in species, a straight line be drawn to the base in a given angle; it shall have a given ratio to the base.

From the vertex A of the triangle ABC which is given in species, let AD be drawn to the base BC in a given angle ADB; the ratio of AD to BC is given.

Because the triangle ABC is given in species, the angle ABD is given, and the angle ADB is given; therefore the triangle ABD is given in species; wherefore the ratio of AD to AB is given, and the ratio of AB to BC is given; and therefore the ratio of AD to BC is given.

PROP. LI.

47. RECTILINEAL figures given in species, are divided into triangles which are given in species.

Let the rectilineal figure ABCDE be given in species; ABCDE may be divided into triangles given in species.

Join BE, BD, and because ABCDE is given in species, the angle BAE is given, and the ratio of BA to BE is given; wherefore the triangle BAE is given in species, and the angle AEB is therefore given. but the whole angle AED is given, and therefore the remaining angle BED is given. and the ratio of AE to EB is given, as also the ratio of AE to ED; therefore the ratio of BE to ED is given. and the angle BED is given, wherefore the triangle BED is given in species. in the same manner the triangle BDC is given in species. therefore rectilineal figures which are given in species are divided into triangles given in species.

PROP. LII.

48. IF two triangles given in species be described upon the same straight line; they shall have a given ratio to one another.

Let the triangles ABC, ABD given in species be described upon the same straight line AB; the ratio of the triangle ABC to the triangle ABD is given.

Thro' the point C draw CE parallel to AB, and let it meet DA produced in E, and join BE. because the triangle ABC is given is species, the angle BAC, that is the angle ACE, is given; and because the triangle ABD is given in species, the angle DAB, that is the angle AEC is given. therefore the triangle ACE is given in species; wherefore the ratio of EA to AC is given, and the ratio of CA to AB is given, as also the ratio of BA to AD; therefore the ratio of EA to AD is given. and the triangle ACB is equal to the triangle AEB, and as the triangle AEB, or ACB, is to the triangle ADB, so is the straight line EA to AD. but the ratio of EA to AD is given, therefore the ratio of the triangle ACB to the triangle ADB is given.

PROBLEM.

To find the ratio of two triangles ABC, ABD given in species and which are described upon the same straight line AB.

Take a straight line FG given in position and magnitude, and because the angles of the triangles ABC, ABD are given, at the points F, G of the straight line FG make the angles GFH, GFK equal to the angles BAC, BAD; and the angles FGH, FGK equal to the angles ABC, ABD, each to each. therefore the triangle ABC, ABD are equiangular to the triangles FGH, FGK, each to each. through the point II draw HL parallel to FG meeting KF produced in L. and because the angles BAC, BAD are equal to the angles GFH, GFK, each to each; therefore the angles ACE, AEC are equal to FHL, FLH, each to each, and the triangle AEC equiangular to the triangle FLH. therefore as EA to AC, so is LF to FH; and as CA to AB, so HF to FG; and as BA to AD, so GF to FK; wherefore, ex aequali, as EA to AD, so is LF to FK. but, as was shewn, the triangle ABC is to the triangle ABD, as the straight line EA to AD, that is as LF to FK. the ratio therefore of LF to FK has been found which is the same with the ratio of the triangle ABC to the triangle ABD.

PROP. LIII.

49. IF two rectilineal figures given in species be described upon the same straight line; they shall have a given ratio to one another.

Let any two rectilineal figures ABCDE, ABFG which are given in species, be described upon the same straight line AB; the ratio of them to one another is given.

Join AC, AD, AF; each of the triangles AED, ADC, ACB, AGF, ABF is given in species. and because the triangles ADE, ADC given in species are described upon the same straight line AD, the ratio of EAD to DAC is given; and, by composition, the ratio of EACD to DAC is given. and the ratio of DAC to CAB is given, because they are described upon the same straight line AC; therefore the ratio of EACD to ACB is given; and, by composition, the ratio of ABCDE to ABC is given. in the same manner, the ratio of ABFG to ABF is given. but the ratio of the triangle ABC to the triangle ABF is given; wherefore because the ratio of ABCDE to ABC is given, as also the ratio of ABC to ABF, and the ratio of ABF to ABFG; the ratio of the rectilineal ABCDE to the rectilineal ABFG is given.

PROBLEM.

To find the ratio of two rectilineal figures given in species, and described upon the same straight line.

Let ABCDE, ABFG be two rectilineal figures given In species, and described upon the same straight line AB, and join AC, AD, AF. take a straight line HK given in position and magnitude, and by the 52. Dat. find the ratio of the triangle ADE to the triangle ADC, and make the ratio of HK to KL the same with it. find also the ratio of the triangle ACD to the triangle ACB, and make the ratio of KL to LM the same. also, find the ratio of the triangle ABC to the triangle ABF, and make the ratio of LM to MN the same. and lastly, find the ratio of the triangle AFB to the triangle AFG, and make the ratio of MN to NO the same. then the ratio of ABCDE to ABFG is the same with the ratio of HM to MO.

Because the triangle EAD is to the triangle DAC, as the straight line HK to KL; and as triangle DAC to CAB, so is the straight line KL to LM; therefore by using composition as often as the number of triangles requires, the rectilineal ABCDE is to the triangle ABC, as the straight line HM to ML. in like manner, because triangle GAF is to FAB, as ON to NM, by composition, the rectilineal ABFG is to the triangle ABF, as MO to MN; and, by inversion, as ABF to ABFG, so is NM to MO. and the triangle ABC is to ABF, as LM to MN. wherefore because as ABCDE to ABC, so is HM to ML; and as ABC to ABF, so is LM to MN; and as ABF to ABFG, so is MN to MO; ex aequali, as the rectilineal ABCDE to ABFG, so is the straight line HM to MO.

PROP. LIV.

50. IF two straight lines have a given ratio to one another; the similar rectilineal figures described upon them similarly, shall have a given ratio to one another.

Let the straight lines AB, CD have a given ratio to one another, and let the similar and similarly placed rectilineal figures E, F be described upon them; the ratio of E to F is given.

To AB, CD let G be a third proportional; therefore as AB to CD, so is CD to G. and the ratio of AB to CD is given, wherefore the ratio of CD to G is given; and consequently the ratio of AB to G is also given. but as AB to G, so is the figure E to the figure F. therefore the ratio of E to F is given.

PROBLEM.

To find the ratio of two similar rectilineal figures E, F similarly described upon straight lines AB, CD which have a given ratio to one another. let G be a third proportional to AB, CD.

Take a straight line H given in magnitude; and because the ratio of AB to CD is given, make the ratio of H to K the same with it; and because H is given, K is given. as H is to K, so make K to L; then the ratio of E to F is the same with the ratio of H to L. for AB is to CD, as H to K, wherefore CD is to G, as K to L; and, ex aequali, as AB to G, so is H to L. but the figure E is to the figure F, as AB to G, that is as H to L.

PROP. LV.

IF two straight lines have a given ratio to one another; the rectilineal figures given in species described upon them, shall have to one another a given ratio.

Let AB, CD be two straight lines which have a given ratio to one another; the rectilineal figures E, F given in species and described upon them, have a given ratio to one another.

Upon the straight line AB describe the figure AG similar and similarly placed to the figure F; and because F is given in species, AG is also given in species. therefore since the figures E, AG which are given in species, are described upon the same straight line AB, the ratio of E to AG is given. and because the ratio of AB to CD is given, and upon them are described the similar and similarly placed rectilineal figures AG, F, the ratio of AG to F is given. and the ratio of AG to E is given; therefore the ratio of E to F is given.

PROBLEM.

To find the ratio of two rectilineal figures E, F given in species, and described upon the straight lines AB, CD which have a given ratio to one another.

Take a straight line H given in magnitude; and because the rectilineal figures E, AG given in species are described upon the same straight line AB, find their ratio by the 53. Dat. and make the ratio of H to K the same; K is therefore given. and because the similar rectilineal figures AG, F are described upon the straight lines AB, CD which have a given ratio, find their ratio by the 54. Dat. and make the ratio of K to L the same. the figure E has to F the same ratio which H has to L. for, by the construction, as E is to AG, so is H to K; and as AG to F, so is K to L; therefore, ex aequali, as E to F, so is H to L.

PROP. LVI.

52. IF a rectilineal figure given in species be described upon a straight line given in magnitude; the figure is given in magnitude.

Let the rectilineal figure ABCDE given in species be described upon the straight line AB given in magnitude; the figure ABCDE is given in magnitude.

Upon AB let the square AF be described; therefore AF is given in species and magnitude. and because the rectilineal figures ABCDE, AF given in species are described upon the same straight line AB, the ratio of ABCDE to AF is given. but the square AF is given in magnitude, therefore also the figure ABCDE is given in magnitude.

PROP.

To find the magnitude of a rectilineal figure given in species described upon a straight line given in magnitude.

Take the straight line GH equal to the given straight line AB, and by the 53. Dat. find the ratio which the square AF upon AB has to the figure ABCDE; and make the ratio of GH to HK the same; and upon GH describe the square GL, and complete the parallelogram LHKM the figure ABCDE is equal to LHKM because AF is to ABCDE, as the straight line GH to HK, that is, as the figure GL to HM; and AF is equal to GL, therefore ABCDE is equal to HM.

PROP. LVII.

IF two rectilineal figures are given in species, and if a side of one of them has a given ratio to a side of the other; the ratios of the remaining sides to the remaining sides shall be given.

Let AC, DF be two rectilineal figures given in species, and let be ratio of the side AB to the side DE be given; the ratios of be remaining sides to the remaining sides are also given.

Because the ratio of AB to DE is given, as also the ratios of AB to BC, and of DE to EF; the ratio of BC to EF is given in the same manner, the ratios of the other sides to the other sides are given.

The ratio which BC has to EF may be found thus; take a straight line G given in magnitude, and because the ratio of BC to BA is given, make the ratio of G to H the same; and because the ratio of AB to DE given, make the ratio of H to K be same; and make the ratio of K to L the same with the given ratio of DE to EF. since therefore as BC to BA, so is G to H; and s BA to DE, so is H to K; and as DE to EF, so is K to L; ex equali, BC is to EF, as G to L. therefore the ratio of G to L as been found which is the same with the ratio of BC to EF.

PROP. LVIII.

IF two similar rectilineal figures have a given ratio to one another; their homologous sides have also a given ratio to one another.

Let the two similar rectilineal figures A, B have a given ratio to one another; their homologous sides have also a given ratio.

Let the side CD be homologous to EF, and to CD, EF let the straight line G be a third proportional, as therefore CD to G, D is the figure A to B; and the ratio of A to B is given, therefore the ratio of CD to G is given; and CD, EF, G are proportionals, wherefore the ratio of CD to EF is given.

The ratio of CD to EF may be round thus; take a straight line H given in magnitude; and because the ratio of the figure A to B is given, make the ratio of H to K the same with it. and, as the 13. Dat. directs to be done, find a mean proportional L between H and K; the ratio of CD to EF is the same with that of H to L. let G be a third proportional to CD, EF; therefore as CD to G, so is (A to B, and so is) H to K. and as CD to EF, so is H to L, as is shewn in the 13. Dat.

PROP. LIX.

54. IF two rectilineal figures given in species have a given ratio to one another; their sides shall likewise have given ratios to one another.

Let the two rectilineal figures A, B given in species have a given ratio to one another; their sides shall also have given ratios to one another.

If the figure A be similar to B, their homologous sides shall have a given ratio to one another, by the preceding Proposition; and because the figures are given in species, the sides of each of them have given ratios to one another. therefore each side of one of them has to each side of the other a given ratio.

But if the figure A be not similar to B, let CD, EF be any two of their sides; and upon EF conceive the figure EG to be described similar and similarly placed to the figure A, so that CD, EF be homologous sides; therefore EG is given in species. and the figure B is given in species, wherefore the ratio of B to EG is given; and the ratio of A to B is given, therefore the ratio of the figure A to EG is given. and A is similar to EG, therefore the ratio of the side CB to EF is given; and consequently the ratios of the remaining sides to the remaining sides are given.

The ratio of CD to EF may be found thus; take a straight H given in magnitude, and because the ratio of the figure A to B given, make the ratio of H to K the same with it. and by the Dat. find the ratio of the figure B to EG, and make the ratio of H to L the same; between H and L find a mean proportional M; ratio of CD to EF is the same with the ratio of H to M. because, the figure A is to B, as H to K; and as B to EG, so is K to L; ex aequali, as A to EG, so is H to L. and the figures A, EG are so b. 9. D and M is a mean proportional between H and L; therefore, as C is shewn in the preceding Proposition, CO is to EF, as H to M.

PROP. LX.

IF a rectilineal figure be given in species and magnitude, the sides of it shall be given in magnitude.

Let the rectilineal figure A be given in species and magnitude; sides are given in magnitude.

Take a straight line BC given in position and magnitude; and upon BC describe the figure D similar, and similarly placed, to the figure A, and let EF be the side of the figure A homologous to BC the side of D; therefore the figure D is given in species. and because upon the given straight line. C the figure D given in species is described, D is given in magnitude. and the figure A is given in magnitude, therefore the ratio of A to D is given. and the figure A is similar to D; therefore the ratio of he side EF to the homologous side BC is given. and BC is given, wherefore EF is given. and the ratio of EF to EG is given, therefore EG is given. and, in the same manner, each of the other sides of the figure A can be shewn to be given.

PROBLEM.

To describe a rectilineal figure A similar to a given figure D, and equal to another given figure H. It is Pro. B. 6. Elem.

Because each of the figures D, H is given, their ratio is given, which may be found by making upon the given straight line BC the parallelogram BK equal to D, and upon its side CK making the parallelogram KL equal to H in the angle KCL equal to the angle MBC. therefore the ratio of D to H, that is of BK to KL is the same with the ratio of BC to CL. and because the figures D, A are similar, and that the ratio of D to A, or H, is the same with the ratio of BC to CL; by the 58. Dat. the ratio of the homologous sides BC, EF is the same with the ratio of BC to the mean proportional between BC and CL. find EF the mean proportional; then EF is the side of the figure to be described, homologous to BC the side of D, and the figure itself can be described by the 18th Prop. B. 6. which, by the construction, is similar to D. and because D is to A, as BC to CL, that is as the figure BK to KL; and that D is equal to BK, therefore A is equal to KL, that is to H.

PROP. LXI.

IF a parallelogram given in magnitude has one of its sides and one of its angles given in magnitude; the other side also is given.

Let the parallelogram ABCD given in magnitude, have the side AB and the angle BAC given in magnitude; the other side AC is given.

Take a straight line EF given in position and magnitude; and because the parallelogram AD is given in magnitude, a rectilineal figure equal to it can be found. and a parallelogram equal to this figure can be applied to the given straight line EF in an angle equal to the given angle BAC. let this be the parallelogram EFGH having the angle FEG equal to the angle BAC. and because the parallelograms AD, EH are equal, and have the angles at A and E equal; the sides about them are reciprocally proportional. therefore is AB to EF, so is EG to AC; and AB, EF, EG are given, therefore also AC is given. Whence the way of finding AC is manifest.

PROP. LXII.

H.

IF a parallelogram has a given angle, the rectangle contained by the sides about that angle has a given ratio to the parallelogram.

Let the parallelogram ABCD have the given angle ABC; the rectangle AB, BC has a given ratio to the parallelogram AC.

From the point A draw AE perpendicular to BC. because the angle ABC is given, as also the angle AEB; the triangle ABE is given in species; therefore the ratio of BA to AE is given. but as BA to AE, so is the rectangle AB, BC to the rectangle AE, BC; therefore the ratio of the rectangle AB, BC to AE, BC, that is to the parallelogram AC is given.

And it is evident how the ratio of the rectangle to the parallelogram may be found, by making the angle FGH equal to the given angle ABC, and drawing, from any point F in one of its sides, FK perpendicular to the other GH; for GF is to FK, as BA to AE, that is, as the rectangle AB, BC to the parallelogram AC.

66. COR. And if a triangle ABC has a given angle ABC, the rectangle AB, BC contained by the sides about that angle, shall have a given ratio to the triangle ABC.

Complete the parallelogram ABCD; therefore, by this Proposition, the rectangle AB, BC has a given ratio to the parallelogram AC; and AC has a given ratio to its half the triangle ABC. therefore the rectangle AB, BC has a given ratio to the triangle ABC.

And the ratio of the rectangle to the triangle is found thus; make the triangle FGK as was shewn in the Proposition; the ratio of GF to the half of the perpendicular FK is the same with the ratio of the rectangle AB, BC to the triangle ABC. because, as was shewn, GF is to FK, as AB, BC to the parallelogram AC; and FK is to its half, as AC is to its half which is the triangle ABC; therefore, ex aequali, GF is to the half of FK, as AB, BC rectangle is to the triangle ABC.

PROP. LXIII.

56. IF two parallelograms be equiangular, as a side of the first to a side of the second, so is the other side of the second to the straight line to which the other side of the first has the same ratio which the first parallelogram has to the second. and consequently if the ratio of the first parallelogram to the second be given, the ratio of the other side of the first to that straight line is given; and if the ratio of the other side of the first to that straight line be given, the ratio of the first parallelogram to the second is given.

Let AC, DF be two equiangular parallelograms, as BC a side of the first is to EF a side of the second, so is DE the other side of the second to the straight line to which AB the other side of the first has the same ratio which AC has to DF.

Produce the straight line AB, and make as BC to EF, so DE to BG, and complete the parallelogram BGHC. therefore because BC, or GH is to EF, as DE to BG, the sides about the equal angles BGH, DEF are reciprocally proportional; wherefore the parallelogram BH is equal to DF. and AB is to BG, as the parallelogram AC is to BH, that is to DF. as therefore BC is to EF, so is DE to BG which is the straight line to which AB the same ratio that AC has to DF.

And if the ratio of the parallelogram AC to DF be given, then the ratio of the straight line AB to BG is given; and if the ratio of AB to the straight line BG be given, the ratio of the parallelogram AC to DF is given.

PROP. LXIV.

74. 73.

IF two parallelograms have unequal, but given, angles, and if as a side of the first to a side of the second, so the other side of the second be made to a certain straight line; if the ratio of the first parallelogram to the second be given, the ratio of the other side of the first to that straight line shall be given. and if the ratio of the other side of the first to that straight line be given, the ratio of the first parallelogram to the second shall be given.

Let ABCD, EFGH be two parallelograms which have the unequal, but given, angles ABC, EFG; and as BC to FG, so make EF to the straight line M. if the ratio of the parallelogram AC to EG be given, the ratio of AB to M is given.

At the point B of the straight line BC make the angle CBK equal to the angle EFG, and complete the parallelogram KBCL. and because the ratio of AC to EG is given, and that AC is equal to the parallelogram KC, therefore the ratio of KC to EG is given; and KC, EG are equiangular; therefore as BC to FG, so is EF to the straight line to which KB has a given ratio, viz. the same which the parallelogram KC has to EG. but as BC to FG, so is EF the straight line M; therefore KB has a given ratio to M

and the ratio of AB to BK is given, because the triangle ABK is given in species. therefore the ratio of AB to M is given.

And if the ratio of AB to M be given, the ratio of the parallelogram AC to EG is given. for since the ratio of KB to BA is given, as also the ratio of AB to M, the ratio of KB to M is given. and because the parallelograms KC, EG are equiangular, as BC to FG, so is EF to the straight line to which KB has the same ratio which the parallelogram KC has to EG. but as BC to FG, so is EF to M. therefore KB is to M, as the parallelogram KC is to EG. and the ratio of KB to M is given, therefore the ratio of the parallelogram KC, that is of AC to EG is given.

75. COR. And if two triangles ABC, EFG have two equal angles, or two unequal, but given, angles ABC, EFG, and if as BC a side of the first to FG a side of the second, so the other side of the second EF be made to a straight line M; if the ratio of the triangles be given, the ratio of the other side of the first to the straight line M is given.

Complete the parallelograms ABCD, EFGH; and because the ratio of the triangle ABC to the triangle EFG is given, the ratio of the parallelogram AC to

EG is given, because the parallelograms are double of the triangles. and because BC is to FG, as EF to M, the ratio of AB to M is given by the 63. Dat. if the angles ABC, EFG are equal; but if they be unequal, but given angles, the ratio of AB to M is given by this Proposition.

And if the ratio of AB to M be given, the ratio of the parallelogram AC to EG is given by the same Propositions; and therefore the ratio of the triangle ABC to EFG is given.

PROP. LXV.

68. IF two equiangular parallelograms have a given ratio to one another, and if one side has to one side a given ratio; the other side shall also have to the other side a given ratio.

Let the two equiangular parallelograms AB, CD have a given ratio to one another, and let the side EB have a given ratio to the side FD; the other side AE has also a given ratio to the other side CF.

Because the two equiangular parallelograms AB, CD have a given ratio to one another; as EB a side of the first is to FD a side of the second, so is FC the other side of the second to the straight line to which AE the other side of the first has the same given ratio which the first parallelogram AB has to the other CD. let this straight line be EG; therefore the ratio of AE to EG is given. and EB is to FD, as FC to EG, therefore the ratio of FC to EG is given, because the ratio of EB to FD is given. and because the ratio of AE to EG, as also the ratio of FC to EG is given; the ratio of AE to CF is given.

The ratio of AE to CF may be found thus; take a straight line H given in magnitude, and because the ratio of the parallelogram AB to CD is given, make the ratio of H to K the same with it. and because the ratio of FD to EB is given, make the ratio of K to L the same, the ratio of AE to CF is the same with the ratio of H to L. make as EB to FD, so FC to EG, therefore, by inversion, as FD to EB, so is EG to FC. and as AE to EG, so is (the parallelogram AB to CD, and so is) H to K; but as EG to FC, so is (FD to EB, and so is) K to L; therefore, ex aequali, as AE to FC, so is H to L.

PROP. LXVI.

69. IF two parallelograms have unequal, but given, angles, and a given ratio to one another; if one side has to one side a given ratio, the other side has also a given ratio to the other side.

Let the two parallelograms ABCD, EFGH which have the given unequal angles ABC, EFG, have a given ratio to one another, and let the ratio of BC to FG be given; the ratio also of AB to EF is given.

At the point B of the straight line BC make the angle CBK equal to the given angle EFG, and complete the parallelogram BKLC. and because each of the angles BAK, AKB is given, the triangle ABK is given in species; therefore the ratio of AB to BK is given. and because, by the hypothesis, the ratio of the parallelogram AC

to EG is given, and that AC is equal to BL; therefore the ratio of BL to EG is given. and because BL is equiangular to EG, and by the hypothesis, the ratio of BC to FG is given; therefore the ratio of KB to EF is given. and the ratio of KB to BA is given; the ratio therefore of AB to EF is given.

The ratio of AB to EF may be found thus; take the straight line MN given in position and magnitude; and make the angle NMO equal to the given angle BAK, and the angle MNO equal to the given angle EFG or AKB. and because the parallelogram BL is equiangular to EG, and has a given ratio to it, and that the ratio of BC to FG is given; find by the 65. Dat. the ratio of KB to EF; and make the ratio of NO to OP the same with it. then the ratio of AB to EF is the same with the ratio of MO to OP. for since the triangle ABK is equiangular to MON, as AB to BK, so is MO to ON; and as KB to EF, so is NO to OP; therefore, ex aequali, as AB to EF, so is MO to OP.

PROP. LXVII.

70. IF the sides of two equiangular parallelograms have given ratios to one another; the parallelograms shall have a given ratio to one another.

Let ABCD, EFGH be two equiangular parallelograms, and let the ratio of AB to EF, as also the ratio of BC to FG be given; the ratio of the parallelogram AC to EG is given.

Take a straight line K given in magnitude, and because the ratio of AB to EF is given, make ratio of K to L the same with it; therefore L is given and because the ratio of BC to FG is given, make the ratio of L to M the same. therefore M is given; and K is given, wherefore the ratio of K to M is given. but the parallelogram AC is to the parallelogram EG, as the straight line K to the straight line M, as is demonstrated in the 23. Prop. of B. 6. Elem. therefore the ratio of AC to EG is given.

From this it is plain how the ratio of two equiangular parallelograms may be found when the ratios of their sides are given.

PROP. LXVIII.

70. IF the sides of two parallelograms which have unequal, but given, angles, have given ratios to one another; the parallelograms shall have a given ratio to one another.

Let two parallelograms ABCD, EFGH which have the given unequal angles ABC, EFG have the ratios of their sides, viz. of AB to EF, and of BC to FG given; the ratio of the parallelogram AC to EG is given.

At the point B of the straight line BC make the angle CBK equal to the given angle EFG, and complete the parallelogram KBCL. and because each of the angles BAK, BKA is given, the triangle ABK is given in species. therefore the ratio of AB to BK is given; and the ratio of AB to EF is given, wherefore the ratio of BK to EF is given. and the ratio of BC to FG is given; and the angle KDC is equal to the angle EFG; therefore the ratio of the parallelogram KC to EG is given. but KC is equal to AC; therefore the ratio of AC to EG is given.

The ratio of the parallelogram AC to EG may be found thus; take the straight line MN given in position and magnitude, and make the angle MNO equal to the given angle KAB, and the angle NMO equal to the given angle AKB or FEH. and because the ratio of AB to EF is given, make the ratio of NO to P the same; also make the ratio of P to Q the same with the given ratio of BC to FG. the parallelogram AC is to EG, as MO to Q.

Because the angle KAB is equal to the angle MNO, and the angle AKB equal to the angle NMO; the triangle AKB is equiangular to NMO. therefore as KB to BA, so is MO to ON; and as BA to EF, so is NO to P; wherefore, ex aequali, as KB to EF, so is MO to P. and BC is to FG, as P to Q, and the parallelograms KC, EG are equiangular; therefore, as was shewn in Pro. the parallelogram KC, that is AC, is to EG, as MO to Q.

COR. 1. If two triangles ABC, DEF have two equal angles, 71. or two unequal, but given angles ABC, DEF, and if the ratios of the sides about these angles, viz. the ratios of AB to DE, and of BC to EF be given; the triangles shall have a given ratio to one another.

Complete the parallelograms BG, EH; the ratio of BG to EH is given; and therefore the triangles which are the halves of them have a given ratio to one another.

COR. 2. If the safes BC, EF of two triangles ABC, DEF have a given ratio to one another, and if also the straight lines AG, DH which are drawn to the bases from the opposite angles, either in equal angles, or unequal, but given, angles AGC, DHF have a given ratio to one another; the triangles shall have a given ratio to one another.

Draw BK, EL parallel to AG, DH, and complete the parallelograms KC, LF. and because the angles AGC, DHF, or their equals the angles KBC, LEF are either equal, or unequal, but given; and that the ratio of AG to DH, that is of KB to LE is given, as also the ratio of BC to EF; therefore the ratio of the parallelogram KC to LF is given. wherefore also the ratio of the triangle ABC to DEF is given.

PROP. LXIX.

61. IF a parallelogram which has a given angle be applied to one side of a rectilineal figure given in species; if the figure have a given ratio to the parallelogram, the parallelogram is given in species.

Let ABCD be a rectilineal figure given in species, and to one side of it AB let the parallelogram ABEF having the given angle ABE be applied; if the figure ABCD has a given ratio to the parallelogram BF, the parallelogram BF is given in species.

Thro' the point A draw AG parallel to BC, and thro' the point C draw CG parallel to AB, and produce G A, CB to the points H,

K. because the angle ABC is given, and the ratio of AB to BC is given, the figure ABCD being given in species; therefore the parallelogram BG is given in species. and because upon the same straight line AB the two rectilineal figures BD, BG given in species are described, the ratio of BD to BG is given. and, by hypothesis, the ratio of BD to the parallelogram BF is given; wherefore the ratio of BF, that is of the parallelogram BH, to BG is given, and therefore the ratio of the straight line KB to BC is given. and the ratio of BC to BA is given, wherefore the ratio of KB to BA is given. and because the angle ABC is given, the adjacent angle ABK is given; and the angle ABE is given, therefore the remaining angle KBE is given. the angle EKB is also given, because it is equal to the angle ABK; therefore the triangle BKE is given in species, and consequently the ratio of EB to BK is given. and the ratio of KB to BA is given, wherefore the ratio of EB to BA is given. and the angle ABE is given, therefore the parallelogram BF is given in species.

A parallelogram similar to BF may be found thus; take a straight line LM given in position and magnitude; and because the angles ABK, ABE are given, make the angle NLM equal to ABK, and the angle NLO equal to ABE and because the ratio of BF to BD is given, make the ratio of LM to P the same with it; and because the ratio of the figure BD to BG is given, find this ratio by the 53. Dat. and make the ratio of P to Q the same. also, because the ratio of CB to BA is given, make the ratio of Q to R the same. and take LN equal to R, thro' the point M draw OM parallel to LN, and complete the parallelogram NLOS; then this is similar to the parallelogram BF.

Because the angle ABK is equal to NLM, and the angle ABE to NLO; the angle KBE is equal to MLO. and the angles BKE, LMO are equal, because the angle ABK is equal to NLM. therefore the triangles BKE, LMO are equiangular to one another, wherefore as BE to BK, so is LO to LM. and because as the figure BF to BD, so is the straight line LM to P; and as BD to BG, so is P to Q; ex aequali, as BF, that is BH, to BG, so is LM to Q. but BH is to BG, as KB to BC; as therefore KB to BC, so is LM to Q. and because BE is to BK, as LO to LM; and as BK to BC, so is LM to Q; and as BC to BA, so Q was made to R; therefore, ex aequali, as BE to BA, so is LO to R, that is to LN. and the angles ABE, NLO are equal; therefore the parallelogram BF is similar to LS.

PROP. LXX.

62. 78.

IF two straight lines have a given ratio to one another, and upon one of them be described a rectilineal figure given in species, and upon the other a parallelogram having a given angle; if the figure have a given ratio to the parallelogram, the parallelogram is given in species.

Let the two straight lines AB, CD have a given ratio to one another, and upon AB let the figure AEB given in species be described, and upon CD the parallelogram DF having the given angle FCD; if the ratio of AEB to DF be given, the parallelogram DF is given in species.

Upon the straight line AB conceive the parallelogram AG to be described similar and similarly placed to FD. and because the ratio of AB to CD is given, and upon them are described the similar rectilineal figures AG, FD; the ratio of AG to FD is given. and the ratio of FD to AEB is given; therefore the ratio of AEB to AG is given; and the angle ABG is given, because it is equal to the angle FCD. because therefore the parallelogram AG which has a given angle ABG is applied to a side AB of the figure AEB given in species, and the ratio of AEB to AG is given, the parallelogram AG is given in species. but FD is similar to AG; therefore FD is given in species.

A parallelogram similar to FD may be found thus; take a straight line H given in magnitude; and because the ratio of the figure AEB to FD is given, make the ratio of II to K the same with it. also because the ratio of the straight line CD to AB is given, find by the 54. Dat. the ratio which the figure FD described upon CD has to the figure AG described upon AB similar to FD; and make the ratio of K to L the same with this ratio, and because the ratios of H to K, and of K to L are given, the ratio of H to L is given

because therefore as AEB to FD, so is H to K; and as FD to AG, so is K to L; ex aequali, as AEB to AG, so is H to L; therefore the ratio of AEB to AG is given. and the figure AEB is given in species, and to its side AB the parallelogram AG is applied in the given angle ABG, therefore by the 69. Dat. a parallelogram may be found similar to AG. let this be the parallelogram MN; MN also is similar to FD. for, by the construction, MN is similar to AG, and AG is similar to FD; therefore the parallelogram FD is similar to MN.

PROP. LXXI.

81. IF the extremes of three proportional straight lines have given ratios to the extremes of other three proportional straight lines; the means shall also have a given ratio to one another. and if one extreme has a given ratio to one extreme, and the mean to the mean; likewise the other extreme shall have to the other a given ratio.

Let A, B, C be three proportional straight lines, and D, E, F three other; and let the ratios of A to D, and of C to F be given. then the ratio of B to E is also given.

Because the ratio of A to D, as also of C to F is given, the ratio of the rectangle A, C to the rectangle D, F is given. but the square of B is equal to the rectangle A, C; and the square of E to the rectangle D, F. therefore the ratio of the square of B to the square of E is given; wherefore also the ratio of the straight line B to E is given.

Next, let the ratio of A to D, and of B to E be given; then the ratio of C to F is also given.

Because the ratio of B to E is given, the ratio of the square of B to the square of E is given. therefore the ratio of the rectangle A, C to the rectangle D, F is given. and the ratio of the side A to the side D is given; therefore the ratio of the other side C to the other F is given.

COR. And if the extremes of four proportionals have to the extremes of four other proportionals given ratios, and one of the means a given ratio to one of the means; the other mean shall have a given ratio to the other mean. as may be shewn in the same manner as in the foregoing Proposition.

PROP. LXXII.

82. IF four straight lines be proportionals; as the first is to the straight line to which the second has a given ratio; so is the third to the straight line to which the fourth has a given ratio.

Let A, B, C, D be four proportional straight lines, viz. as A to B, so C to D; as A is to the straight line to which B has a given ratio, so is C to the straight line to which D has a given ratio.

Let E be the straight line to which B has a given ratio, and as B to E, so make D to F. the ratio of B to E is given, and therefore the ratio of D to F. and because as A to B, so is C to D; and as B to E, so D to F; therefore, ex aequali, as A to E, so is C to F. and E is the straight line to which B has a given ratio, and F that to which D has a given ratio; therefore as A is to the straight line to which B has a given ratio, so is C to that to which D has a given ratio.

PROP. LXXIII.

83. IF four straight lines be proportionals; as the first is to the straight line to which the second has a given ratio, so is the straight line to which the third has a given ratio to the fourth.

Let the straight line A be to B, as C to D; as A to the straight line to which B has a given ratio, so is the straight line to which C has a given ratio to D.

Let E be the straight line to which B has a given ratio, and as B to E, so make F to C; because the ratio of B to E is given, the ratio of C to F is given. and because A is to B, as C to D; and as B to E, so F to C; therefore, ex aequali in proportione perturbata, A is to E, as F to D; that is A is to E to which B has a given ratio, as F, to which C has a given ratio, is to D.

PROP. LXXIV.

IF a triangle has a given obtuse angle; the excess of the square of the side which subtends the obtuse angle above the squares of the sides which contain it, shall have a given ratio to the triangle.

Let the triangle ABC have a given obtuse angle ABC; and produce the straight line CB, and from the point A draw AD perpendicular to BC. the excess of the square of AC above the squares of AB, BC, that is the double of the rectangle contained by DB, BC, has a given ratio to the triangle ABC.

Because the angle ABC is given, the angle ABD is also given; and the angle ADB is given, wherefore the triangle ABD is given in species; and therefore the ratio of AD to DB is given. and as AD to DB, so is the rectangle AD, BC to the rectangle DB, BC; wherefore the ratio of the rectangle AD, BC to the rectangle DB, BC is given, as also the ratio of twice the rectangle DB, BC to the rectangle AD, BC. but the ratio or the rectangle AD, BC to the triangle ABC is given, because it is double of the triangle; therefore the ratio of twice the rectangle DB, BC to the triangle ABC is given. and twice the rectangle DB, BC is the excess of the square of AC above the squares of AB, BC. therefore this excess has a given ratio to the triangle ABC.

And the ratio of this excess to the triangle ABC may be found thus; take a straight line EF given in position and magnitude; and because the angle ABC is given, at the point F of the straight line EF make the angle EFG equal to the angle ABC; produce GF, and draw EH perpendicular to FG. then the ratio of the excess of the square of AC above the squares of AB, BC to the triangle ABC is the same with the ratio of quadruple the straight line HF to HE.

Because the angle ABD is equal to the angle EFH, and the angle ADB to EHF, each being a right angle; the triangle ADB is equiangular to EHF. therefore as BD to DA, so FH to HE; and as quadruple of BD to DA, so is quadruple of

FH to HE. but as twice BD is to DA, so is twice the rectangle DB, BC to the rectangle AD, BC; and as DA to the half of it, so is the rectangle AD, BC to its half the triangle ABC; therefore, ex ae quali, as twice BD is to the half of DA, that is, as quadruple of BD is to DA, that is, as quadruple of FH to HE, so is twice the rectangle DB, BC to the triangle ABC.

PROP. LXXV.

65. IF a triangle has a given acute angle; the space by which the square of the side subtending the acute angle is less than the squares of the sides which contain it, shall have a given ratio to the triangle.

Let the triangle ABC have a given acute angle ABC, and draw AD perpendicular to BC; the space by which the square of AC is less than the squares of AB, BC, that is the double of the rectangle contained by CB, BD, has a given ratio to the triangle ABC.

Because the angles ABD, ADB are each of them given, the triangle ABD is given in species; and therefore the ratio of BD to DA is given. and as BD to DA, so is the rectangle CB, BD to the rectangle CB, AD; therefore the ratio of these rectangles is given, as also the ratio of twice the rectangle CB, BD to the rectangle CB, AD. but the rectangle CB, AD has a given ratio to its half the triangle ABC, therefore the ratio of twice the rectangle CB, BD to the triangle ABC is given. and twice the rectangle CB, BD is the space b which the square of AC is less than the squares of AB, BC; therefore the ratio of this space to the triangle ABC is given. and the ratio may be found as in the preceding Proportion.

LEMMA.

IF from the vertex A of an Isosceles triangle ABC, any straight line AD be drawn to the base BC; the square of the side AB is equal to the rectangle BD, DC of the segments of the base together with the square of AD. but if AD be drawn to the base produced, the square of AD is equal to the rectangle BD, DC together with the square of AB.

CAS. 1. Bisect the base BC in E, and join AE which will be perpendicular to BC; wherefore the square of AB is equal to the squares of AE, EB. but the square of EB is equal to the rectangle BD, DC together with the square of DE. therefore the square of AB is equal to the squares of AE, ED, that is to the square of AD, together with the rectangle BD, DC. the other case is shewn in the same way by 6. 2. Elem.

PROP. LXXVI.

67. IF a triangle have a given angle, the excess of the square of the straight line which is equal to the two sides that contain the given angle, above the square of the third side, shall have a given ratio to the triangle.

Let the triangle ABC have the given angle BAC, the excess of the straight line which is equal to BA, AC together above the square of BC, shall have a given ratio to the triangle ABC.

Produce BA, and take AD equal to AC, join DC and produce it to E, and thro' the point B draw BE parallel to AC; join AE, and draw AF perpendicular to DC. and because AD is equal to AC, BD is equal to BE; and BC is drawn from the vertex B of the Isosceles triangle DBE, therefore, by the Lemma, the square of BD, that is of BA and AC together, is equal to the rectangle DC, CE together with the square of BC; and therefore the square of BA, AC together, that is of BD is greater than the square of BC by the rectangle DC, CE; and this rectangle has a given ratio to the triangle ABC. because the angle BAC is given, the adjacent angle CAD is given; and each of the angles ADC, DCA is given, for each of them is the half of the given angle BAC; therefore the triangle ADC is given in species; and AF is drawn from its vertex to the base in a given angle, wherefore the ratio of AF to the base CD is given. and as CD to AF, so is the rectangle DC, CE to the rectangle AF, CE; and the ratio of the rectangle AF, CE to its half the triangle ACE is given; therefore the ratio of the rectangle DC, CE to the triangle ACE, that is to the triangle ABC is given. and the rectangle DC, CE is the excess of the square of BA, AC together above the square of BC; therefore the ratio of this excess to the triangle ABC is given.

The ratio which the rectangle DC, CE has to the triangle ABC is found thus. take the straight line GH given in position and magnitude, and at the point G in GH make the angle HGK equal to the given angle CAD, and take GK equal to GH, join KH, and draw GL perpendicular to it. then the ratio of HK to the half of GL is the same with the ratio of the rectangle DC, CE to the triangle ABC. because the angles HGK, DAC at the vertices of the Isosceles triangles GHK, ADC are equal to one another, these triangles are similar, and because GL, AF are perpendicular to the bases HK, DC, as HK to GL, so is (DC to AF, and so is) the rectangle DC, CE to the rectangle AF, CE; but as GL to its half, so is the rectangle AF, CE to its half which is the triangle ACE, or the triangle ABC; therefore, ex aequali, HK is to the half of the straight line GL, as the rectangle DC, CE is to the triangle ABC.

COR. And if a triangle have a given angle, the space by which the square of the straight line which is the difference of the sides which contain the given angle is less than the square of the third side, shall have a given ratio to the triangle. this is demonstrated the same way as the preceding Proposition, by help of the second case of the Lemma.

PROP. LXXVII.

I.

IF the perpendicular drawn from a given angle of a triangle to the opposite side, or base, has a given ratio to the base; the triangle is given in species.

Let the triangle ABC have the given angle BAC, and let the perpendicular AD drawn to the base BC, have a given ratio to it; the triangle ABC is given in species.

If abc be an Isosceles triangle, it is evident that if any one of its angles be given, the rest are also given; and therefore the triangle

is given in species, without the consideration of the ratio of the perpendicular to the base, which in this case is given by Pro. But when ABC is not an Isosceles triangle, take any straight line EF given in position and magnitude, and upon it describe the segment of a circle EGF containing an angle equal to the given angle BAC; draw GH bisecting EF at right angles, and join EG, GF. then since the angle EGF is equal to the angle BAC, and that EGF is an Isosceles triangle and ABC is not, the angle FEG is not equal to the angle CBA. draw EL making the angle FEL equal to the angle CBA, join FL, and draw LM perpendicular to EF. then because the triangles ELF, BAC are equiangular, as also are the triangles MLE, DAB, as ML to LE, so is DA to AB; and as LE to EF, so is AD to BC; wherefore, ex aequali, as LM to EF, so is AD to BC. and because the ratio of AD to BC is given, therefore the ratio of LM to EF is given; and EF is given, wherefore LM also is given complete the parallelogram LMFK, and because LM is given, FK is given in magnitude; it is also given in position, and the point F is given, and consequently the point K; and because thro’ K the straight line KL is drawn parallel to EF which is given in position, therefore KL is given in position; and the circumference ELF is given in position, therefore the point L is given. and because the points L, E, F are given, the straight lines LE, EF, FL are given in magnitude; therefore, the triangle LEF is given in species. and the triangle ABC is similar to LEF, wherefore also ABC is given in species.

Because LM is less than GH, the ratio of LM to EF, that is the given ratio of AD to BC must be less than the ratio of GH to EF which the straight line, in a segment of a circle containing an angle equal to the given angle, that bisects the base of the segment at right angles, has unto the base.

COR. 1. If two triangles ABC, LEF have one angle BAC equal to one angle ELF, and if the perpendicular AD be to the base BC, as the perpendicular LM to the base EF; the triangles ABC, LEF are similar.

Describe the circle EGF about the triangle ELF, and draw LN parallel to EF, join EN, NF, and draw NO perpendicular to EF. because the angles ENF, ELF are equal, and that the angle EFN is equal to the alternate angle FNL, that is to the angle FEL in the same segment, therefore the triangle NEF is similar to LEF. and in the segment EGF there can be no other triangle upon the base EF which has the ratio of its perpendicular to that base the same with the ratio of LM or NO to EF, because the perpendicular must be greater or less than LM or NO. but, as has been shewn in the preceding demonstration, a triangle similar to ABC can be described in the segment EGF upon the base EF, and the ratio of its perpendicular to the base is the same, as was there shewn, with the ratio of AD to BC, that is of LM to EF. therefore that triangle must be either LEF, or NEF, which therefore are similar to the triangle ABC.

COR. 2. If a triangle ABC has a given angle BAC, and if the straight line AR drawn from the given angle to the opposite side BC, in a given angle ARC, has a given ratio to BC; the triangle ABC is given in species.

Draw AD perpendicular to BC; therefore the triangle ARD is given in species; wherefore the ratio of AD to AR is given; and the ratio of AR to BC is given, and consequently the ratio of AD to BC is given; and the triangle ABC is therefore given in species.

COR. 3. If two triangles ABC, LEF have one angle BAC equal to one angle ELF, and if straight lines drawn from these angles to the bases, making with them given and equal angles, have the same ratio to the bases, each to each; then the triangles are similar. for, having drawn perpendiculars to the bases from the equal angles, as one perpendicular is to its base, so is the other to its base. wherefore, by Cor. 1. the triangles are similar.

A triangle similar to ABC may be found thus; having described the segment EGF and drawn the straight line GH as was directed in the Proposition, find FK which has to EF the given ratio of AD to BC; and place FK at right angles to EF from the point F. then because, as has been shewn, the ratio of AD to BC, that is of FK to EF, must be less than the ratio of GH to EF; therefore FK is less than GH; and consequently the parallel to EF drawn through the point K must meet the circumference of the segment in two points. let L be either of them, and join EL, LF, and draw LM perpendicular to EF. then because the angle BAC is equal to the angle ELF, and that AD is to BC, as KF, that is LM to EF, the triangle ABC is similar to triangle LEF. by Cor. 1.

PROP. LXXVIII.

80. IF a triangle have one angle given, and if the ratio of the rectangle of the sides which contain the given angle to the square of the third side be given; the triangle is given in species.

Let the triangle ABC have the given angle BAC, and let the ratio of the rectangle BA, AC to the square of BC be given; the triangle ABC given in species.

From the point A draw AD perpendicular to BC; the rectangle AD, BC has a given ratio to its half the triangle ABC. and because the angle BAC is given, the ratio of the triangle ABC to the rectangle BA, AC is given; and, by the hypothesis, the ratio of the rectangle BA, AC to the square of BC is given. therefore the ratio of the rectangle AD, BC to the square of BC, that is the ratio of the straight line AD to BC is given. wherefore the triangle ABC is given in species.

A triangle similar to ABC may be found thus; take a straight line EF given in position and magnitude, and make the angle FEG equal to the given angle BAC, and draw FH perpendicular to EG, and BK perpendicular to AC; therefore the triangles ABK, EFH are similar. and the rectangle AD, BC, or the rectangle BK, AC, which is equal to it, is to the rectangle BA, AC, as the straight line BK to BA, that is as FH to FE; let the given ratio of the rectangle BA, AC to the square of BC be the same with the ratio of the straight line EF to FL; therefore, ex aequali, the radio of the rectangle AD, BC to the square of

BC, that is the ratio of the straight line AD to BC, is the same with the ratio of HF to FL. and because AD is not greater than the straight line MN in the segment of the circle described about the triangle ABC, which bisects BC at right angles; the ratio of AD to BC, that is of HF to FL, must not be greater than the ratio of MN to BC. let it be so, and by the 77. Dat. find a triangle OPQ which has one of its angles POQ equal to the given angle BAC, and the ratio of the perpendicular OR, drawn from that angle, to the base PQ the same with the ratio of HF to FL. then the triangle ABC is similar to OPQ. because, as has been shewn, the ratio of AD to BC is the same with the ratio of (HF to FL, that is, by the construction, with the ratio of) OR to PQ; and the angle BAC is equal to the angle POQ. therefore the triangle ABC is similar to the triangle POQ.

Otherwise,

Let the triangle ABC have the given angle BAC, and let the ratio of the rectangle BA, AC to the square of BC be given; the triangle ABC is given in species.

Because the angle BAC is given, the excess of the square of both the sides BA, AC together above the square of the third side BC has a given ratio to the triangle ABC. let the figure D be equal to this excess; therefore the ratio of D to the triangle ABC is given; and the ratio of the triangle ABC to the rectangle BA, AC is given, because BAC is a given angle; and the rectangle BA, AC has a given ratio to the square of BC; wherefore the ratio of D to the square of BC is given. and, by composition, the ratio of the space D together with the square of BC to the square of BC is given. but D together with the square of BC is equal to the square of both BA and AC together; therefore the ratio of the square of BA, AC together to the square of BC is given; and the ratio of BA, AC together to BC is therefore given. and the angle BAC is given, wherefore the triangle ABC is given in species.

The composition of this which depends upon those of the 76. and 48. Propositions is more complex than the preceding composition which depends upon that of Pro. which is easy.

PROP. LXXIX.

K.

IF a triangle have a given angle, and if the straight line drawn from that angle to the base, making a given angle with it, divides the base into segments which have a given ratio to one another; the triangle is given in species.

Let the triangle ABC have the given angle BAC, and let the straight line AD drawn to the base BC making the given angle ADB, divide BC into the segments BD, DC which have a given ratio to one another; the triangle ABC is given in species.

Describe the circle BAC about the triangle, and from its center E draw EA, EB, EC, ED. because the angle BAC is given, the angle BEC at the center, which is the double of it, is given. and the ratio of BE to EC is given, because they are equal to one another; therefore the triangle BEC is given in species, and the ratio of EB to BC is given. also the ratio of CB to BD is given, because the ratio of BD to DC is given; therefore the ratio of EB to BD is given. and the angle EBC is given, wherefore the triangle EBD is given in species, and the ratio of EB, that is of EA to ED is therefore given. and the angle EDA is given, because each of the angles BDE, BDA is given. therefore the triangle AED is given in species, and the angle AED given; also the angle DEC is given, because each of the angles BED, BEC is given; therefore the angle AEC is given. and the ratio of EA to EC, which are equal, is given; and the triangle AEC is therefore given in species, and the angle ECA given. and the angle ECB is given, wherefore the angle ACB is given. and the angle BAC is also given; therefore the triangle ABC is given in species.

A triangle similar to ABC may be found, by taking a straight line given in position and magnitude, and dividing it in the given ratio which the segments BD, DC are required to have to one another; then if upon that straight line a segment of a circle be described containing an angle equal to the given angle BAC, and a straight line be drawn from the point of division in an angle equal to the given angle ADB, and from the point where it meets the circumference, straight lines be drawn to the extremity of the first line, these together with the first line shall contain a triangle similar to ABC, as may easily be shewn.

The Demonstration may be also made in the manner of that of the 77. Prop. and that of the 77. may be made in the manner of this.

PROP. LXXX.

L.

IF the sides about an angle of a triangle have a given ratio to one another, and if the perpendicular drawn from that angle to the base has a given ratio to the base; the triangle is given in species.

Let the sides BA, AC, about the angle BAC of the triangle ABC have a given ratio to one another, and let the perpendicular AD have a given ratio to the base BC; the triangle ABC is given in species.

First, let the sides AB, AC be equal to one another, therefore the perpendicular AD bisects the base BC. and the ratio of AD to BC, and therefore to its half DB is given; and the angle ADB is given. wherefore the triangle* ABD and consequently the triangle ABC is given in species.

But let the sides be unequal, and BA be greater than AC; and make the angle CAE equal to the angle ABC. because the angle AEB is common to the triangles AEB, CEA, they are similar; therefore as AB to BE, so is CA to AE, and, by permutation, as BA to AC, so is BE to EA, and so is EA to EC. and the ratio of BA to AC is given, therefore the ratio of BE to EA, and the ratio of E A to EC, as also the ratio of BE to EC is given; wherefore the ratio of EB to BC is given. and the ratio of AD to BC is given by the Hypothesis, therefore the ratio of AD to BE is given; and the ratio of BE to EA was shewn to be given; wherefore the ratio of AD to AE is given. and ADE is a right angle, therefore the triangle ADE is given in species, and the angle AEB given; the ratio of BE to EA is likewise given, therefore the triangle ABE is given in species, and consequently the angle EAB, as also the angle ABE, that is the angle CAE is given; therefore the angle BAC is given, and the angle ABC being also given, the triangle ABC is given in species.

How to find a triangle which shall have the things which are mentioned to be given in the Proposition, is evident in the first case. and to find it the more easily in the other case, it is to be observed that if the straight line EF equal to EA be placed in EB towards B, the point F divides the base BC into the segments BF, FC which have to one another the ratio of the sides BA, AC. because BE, EA, or EF, and EC were shewn to be proportionals, therefore* BF is to FC, as BE to EF, or EA, that is as BA to AC. and AE cannot be less than the altitude of the triangle ABC, but it may be equal to it; which if it be, the triangle, in this case, as also the ratio of the sides, may be thus found, having given the ratio of the perpendicular to the base. take the straight line GH given in position and magnitude, for the base of the triangle to be found; and let the given ratio of the perpendicular to the base be that of the straight line K to GH, that is, let K be equal to the perpendicular; and suppose GLH to be the triangle which is to be found. therefore having made the angle HLM equal to LGH, it is required that LM be perpendicular to GM and equal to K. and because GM. ML, MH are proportionals, as was shewn of BE, EA, EC, the rectangle GMH is equal to the square of ML. add the common square of NH, (having bisected GH in N) and the square of NM is equal to the squares of the given straight lines NH and ML, or K. therefore the square of NM, and its side NM, is given, as also the point M, viz. by taking the straight line NM the square of which is equal to the squares of NH, ML. draw ML equal to K, at right angles to GM. and because ML is given in position and magnitude, therefore the point L is given; join LG, LH, then the triangle LGH is that which was to be found. for the square of NM is equal to the squares of NH and ML, and taking away the common square of NH, the rectangle GMH is equal to the square of ML; therefore as GM to ML, so is ML to MH, and the triangle LGM is therefore equiangular to HLM, and the angle HLM equal to the angle LGM, and the straight line LM, drawn from the vertex of the triangle making the angle HLM equal to LGH, is perpendicular to the base and equal to the given straight line K, as was required. and the ratio of the sides GL, LH is the same with the ratio of GM to ML, that is with the ratio of the straight line which is made up of GN the half of the given base and of NM the square of which is equal to the squares of GN and K, to the straight line K.

And whether this ratio of GM to ML is greater or less than the ratio of the sides of any other triangle upon the base GH, and of which the altitude is equal to the straight line K, that is, the vertex of which is in the parallel to GH drawn thro’ the point L, may be thus found. Let OGH be any such triangle, and draw OP making the angle HOP equal to the angle OGH; therefore, as before, GP, PO, PH are proportionals. and PO cannot be equal to LM, because the rectangle GPH, would be equal to the rectangle GMH, which is impossible, for the point P cannot fall upon M, because O would then fall on L; nor can PO be less than LM, therefore it is greater; and consequently the rectangle GPH is greater than the rectangle GMH, and the straight line GP greater than GM. therefore the ratio of GM to MH is greater than the ratio of GP to PH, and the ratio of the square of GM to the square of ML is therefore greater than the ratio of the square of GP to the square of PO, and the ratio of the straight line GM to ML, greater than the ratio of GP to PO. but as GM to ML, so is GL to LH; and as GP to PO, so is GO to OH; therefore the ratio of GL to LH is greater than the ratio of GO to OH; wherefore the ratio of GL to LH is the greatest of all others; and consequently the given ratio of the greater side to the less must not be greater than this ratio.

But if the ratio of the sides be not the same with this greatest ratio of GM to ML, it must necessarily be less than it. Let any less ratio be given, and the same things being supposed, viz. that GH is the base, and K equal to the altitude of the triangle, it may be found as follows. Divide GH in the point Q, so that the ratio of GQ to QH may be the same with the given radio of the sides; and as GQ to QH, so make GP to PQ, and so will PQ be to PH; wherefore the square of GP is to the square of PQ, as the straight line GP to PH. and because GM, ML, MH are proportionals, the square of GM is to the square of ML, as the straight line GM to MH. but the ratio of GQ to QH, that is the ratio of GP to PQ, is less than the ratio of GM to ML; and therefore the ratio of the square of GP to the square of PQ is less than the ratio of the square of GM to that of ML; and consequently the ratio of the straight line GP to PH is less than the ratio of GM to MH, and, by division, the ratio of GH to HP is less than that of GH to HM; wherefore k the straight line HP is greater than HM, and the rectangle GPH, that is the square of PQ. greater than the rectangle GMH, that is than the square of ML, and the straight line PQ is therefore greater than ML. draw LR parallel to GP, and from P draw PR at right angles to GP. because PQ is greater than ML, or PR, the circle described from the center P, at the distance PQ, must necessarily cut LR in two points; let these be O, S, and join OG, OH; SG, SH; each of the triangles OGH, SGH have the things mentioned to be given in the Proposition. join OP, SP; and because as GP to PQ, or PO, so is PO to PH, the triangle OGP is equiangular to HOP; as, therefore, OG to GP, so is HO to OP, and, by permutation, as GO to OH, so is GP to PO, or PQ, and so is GQ to QH. therefore the triangle OGH has the ratio of its sides GO, OH the same with the given ratio of GQ to QH; and the perpendicular has to the base the given ratio of K to GH, because the perpendicular is equal to LM, or K. the like may be shewn in the same way of the triangle SGH.

This construction by which the triangle OGH is found, is shorter than that which would be deduced from the Demonstration of the Datum; by reason that the base GH is given in position and magnitude, which was not supposed in the Demonstration. the same thing is to be observed in the next Proposition.

PROP. LXXXI.

IF the sides about an angle of a triangle be unequal and have a given ratio to one another, and if the perpendicular from that angle to the base divides it into segments that have a given ratio to one another; the triangle is given in species.

Let ABC be a triangle the sides of which about the angle BAC are unequal, and have a given ratio to one another, and let the perpendicular AD to the base BC divide it into the segments BD, DC which have a given ratio to one another; the triangle ABC is given in species.

Let AB be greater than AC, and make the angle CAE equal to the angle ABC; and because the angle AEB is common to the triangles ABE, CAE, they are equiangular to one another. therefore as AB to BE, so is CA to AE, and, by permutation, as AB to AC, so BE to EA, and so is EA to EC. but the ratio of BA to AC is given, therefore the ratio of BE to EA, as also the ratio of EA to EC is given; wherefore the ratio of BE to EC, as also the ratio of EC to CB is given. and the ratio of BC to CD is given, because the ratio of BD to DC is given; therefore the ratio of EC to CD is given, and consequently the ratio of DE to EC. and the ratio of EC to EA was shewn to be given, therefore the ratio of DE to EA is given. and ADE is a right angle, wherefore the triangle ADE is given in species, and the angle AED given. and the ratio of CE to EA is given, therefore the triangle AEC is given in species, and consequently the angle ACE is given, as also the adjacent angle ACB. in the same manner, because the ratio of BE to EA is given, the triangle BEA is given in species, and the angle ABE is therefore given. and the angle ACB is given; wherefore the triangle ABC is given in species.

But the ratio of the greater side BA to the other AC must be less than the ratio of the greater segment BD to DC. because the square of BA is to the square of AC, as the squares of BD, DA to the squares of DC, DA; and the squares of BD, DA have to the squares of DC, DA a less ratio than the square of BD has to the square of DC, because the square of BD is greater than the square of DC; therefore the square of BA has to the square of AC a less ratio than the square of BD has to that of DC. and consequently the ratio of BA to AC is less than the ratio of BD to DC.

This being premised, a triangle which shall have the things mentioned to be given in the Proposition, and to which the triangle ABC is similar, may be found thus. take a straight line GH given in position and magnitude, and divide it in K so that the ratio of GK to KH may be the same with the given ratio of BA to AC; divide also GH in L so that the ratio of GL to LH may be the same with the given ratio of BD to DC, and draw LM at right angles to GH. and because the ratio of the sides of a triangle is less than the ratio of the segments of the base, as has been shewn, the ratio of GK to KH is less than the ratio of GL to LH, wherefore the point L must fall betwixt K and H. also make as GK to KH, so GN to NK, and so shall NK be to NH. and from the center N, at the distance NK describe a circle, and let its circumference meet LM in O, and join OG, OH; then OGH is the triangle which was to be described. because GN is to NK, or NO, as NO to NH, the triangle OGN is equiangular to HON; therefore as OG to GN, so is HO to ON, and, by permutation, as GO to OH, so is GN to NO, or NK, that is as GK to KH, that is in the given ratio of the sides. and, by the construction, GL, LH have to one another the given ratio of the segments of the base.

PROP. LXXXII.

60. IF a parallelogram given in species and magnitude be encreased, or diminished by a gnomon given in magnitude; the sides of the gnomon are given in magnitude.

First, let the parallelogram AB given in species and magnitude be encreased by the given gnomon ECBDFG; each of the straight lines CE, DF is given.

Because AB is given in species and magnitude, and that the gnomon ECBDFG is given, therefore the whole space AG is given in magnitude. but AG is also given in species, because it is similar to AB; therefore the sides of AG are given. each of the straight lines AE, AF is therefore given; and each of the straight lines CA, AD is given, therefore each of the remainders EC, DF is given.

Next, let the parallelogram AG given in species and magnitude be diminished by the given gnomon ECBDFG; each of the straight lines CE, DF is given.

Because the parallelogram AG is given, as also its gnomon ECBDFG; the remaining space AB is given in magnitude. but it is also given in species; because it is similar to AG; therefore its sides CA, AD are given. and each of the straight lines EA, AF is given; therefore EC, DF are each of them given.

The gnomon and its sides CE, DF may be found thus in the first case. let H be the given space to which the gnomon must be made equal, and find a parallelogram similar to AB and equal to the figures AB and H together, and place its sides AE, AF from the point A, upon the straight lines AC, AD, and complete the parallelogram AG, which is about the same diameter with AB. because therefore AG is equal to both AB and H, take away the common part AB, the remaining gnomon ECBDFG is equal to the remaining figure H. therefore a gnomon equal to H, and its sides CE, DF are found. and in like manner they may be found in the other case, in which the given figure H must be less than the figure FE from which it is to be taken.

PROP. LXXXIII.

58. IF a parallelogram equal to a given space be applied to a given straight line, deficient by a parallelogram given in species; the sides of the defect are given.

Let the parallelogram AC equal to a given space be applied to the given straight line AB, deficient by the parallelogram BDCL given in species; each of the straight lines CD, DB are given.

Bisect AB in E; therefore EB is given in magnitude. upon EB describe the parallelogram EF similar to DL and similarly placed; therefore EF is given in species, and is about the same diameter with DL; let BCG be the diameter, and construct the figure. therefore because the figure EF given in species is described upon the given straight line EB, EF is given in magnitude. and the gnomon ELH is equal to the given figure AC, therefore since EF is diminished by the given gnomon ELH, the sides EK, FH of the gnomon are given. but EK is equal to DC, and FH to DB; wherefore CD, DB are each of them given.

This Demonstration is the Analysis of the Problem in the 28. Prop. of Book 6. the construction and Demonstration of which Proposition is the Composition of the Analysis. and because the given space AC or its equal the gnomon ELH is to be taken from the figure EF described upon the half of AB similar to BC, therefore AC must not be greater than EF, as is shewn in the 27. Prop. B. 6.

PROP. LXXXIV.

59. IF a parallelogram equal to a given space be applied to a given straight line, exceeding by a parallelogram given in species; the sides of the excess are given.

Let the parallelogram AC equal to a given space be applied to the given straight line AB, exceeding by the parallelogram BDCL given in species; each of the straight lines CD, DB are given.

Bisect AB in E; therefore EB is given in magnitude. upon EB describe the parallelogram EF similar to LD, and similarly placed; therefore EF is given in species, and is about the same diameter

with LD. let CBG be the diameter, and construct the figure. therefore because the figure EF given in species is described upon the given straight line EB, EF is given in magnitude. and the gnomon ELH is and equal to the given figure AC; wherefore since EF is encreased by the given gnomon ELH, its sides EK, FH are given. but EK is equal to CD, and FH to BD; therefore CD, DB are each of them given.

This Demonstration is the Analysis of the Problem in the 29. Prop. Book 6. the construction and Demonstration of which is the Composition of the Analysis.

COR. If a parallelogram given in species be applied to a given straight line, exceeding by a parallelogram equal to a given space; the sides of the parallelogram are given.

Let the parallelogram ADCE given in species be applied to the given straight line AB exceeding by the parallelogram BDCG equal to a given space; the sides AD, DC of the parallelogram are given.

Draw the diameter DE of the parallelogram AC, and construct the figure. because the parallelogram AK is equal to BC which is given, therefore AK is given. and BK is similar to AC, therefore BK is given in species. and since the parallelogram AK given in magnitude is applied to the given straight line AB, exceeding by the parallelogram BK given in species, therefore, by this Proposition, BD, DK the sides of the excess are given. and the straight line AB is given, therefore the whole AD, as also DC to which it has a given ratio is given.

PROB.

To apply a parallelogram similar to a given one to a given straight line AB, exceeding by a parallelogram equal to a given space.

To the given straight line AB apply the parallelogram AK equal to the given space, exceeding by the parallelogram BK similar to the one given. draw DF the diameter of BK, and thro' the point A draw AE parallel to BF meeting DF produced in E, and complete the parallelogram AC.

The parallelogram BC is equal to AK, that is to the given space; and the parallelogram AC is similar to BK. therefore the parallelogram AC is applied to the straight line AB similar to the one given and exceeding by the parallelogram BC which is equal to the given space.

PROP. LXXXV.

34. IF two straight lines contain a parallelogram given in magnitude, in a given angle; if the difference of the straight lines be given, they shall each of them be given.

Let AB, BC contain the parallelogram AC given in magnitude, in the given angle ABC, and let the excess of BC above AB be given; each of the straight lines AB, BC is given.

Let DC be the given excess of BC above BA, therefore the remainder BD is equal to BA. complete the parallelogram AD, and because AB is equal to BD, the ratio of AB to BD is given. and the angle ABD is given, therefore the parallelogram AD is given in species. and because the given parallelogram AC is applied to the given straight line DC, exceeding by the parallelogram AD given in species, the sides of the excess are given; therefore BD is given. and DC is given, wherefore the whole BC is given. and AB is given, therefore AB, BC are each of them given.

PROP. LXXXVI.

85. IF two straight lines contain a parallelogram given in magnitude, in a given angle; if both of them together be given, they shall each of them be given.

Let the two straight lines AB, BC contain the parallelogram AC given in magnitude, in the given angle ABC, and let AB, BC together be given; each of the straight lines AB, BC is given.

Produce CB and make BD equal to BA, and complete the parallelogram ABDE. because DB is equal to BA, and the angle ABD given, because the adjacent angle ABC is given; the parallelogram AD is given in species. and because AB, BC together are given, and AB is equal to BD; therefore DC is given. and because the given parallelogram AC is applied to the given straight line DC, deficient by the parallelogram AD given in species, the sides AB, BD of the defect are given. and DC is given, wherefore the remainder BC is given; and each of the straight lines AB, BC is therefore given.

PROP. LXXXVII.

87. IF two straight lines contain a parallelogram given in magnitude, in a given angle; if the excess of the square of the greater above the square of the lesser be given, each of the straight lines shall be given.

Let the two straight lines AB, BC contain the given parallelogram AC in the given angle ABC; if the excess of the square of BC above the square of BA be given; AB and BC are each of them given.

Let the given excess of the square of BC above the square of BA be the rectangle CB, BD; take this from the square of BC, the remainder, which is the rectangle BC, CD is equal to the square of AB. and because the angle ABC of the parallelogram AC is given, the ratio of the rectangle of the sides AB, BC to the parallelogram AC is given; and AC is given, therefore the rectangle AB, BC is given; and the rectangle CB, BD is given; therefore the ratio of the rectangle CB, BD to the rectangle AB, BC, that is the ratio of the straight line DB to BA is given; therefore the ratio of the square of DB to the square of BA is given. and the square of BA is equal to the rectangle BC, CD; wherefore the ratio of the rectangle BC, CD to the square of BD is given, as also the ratio of four times the rectangle BC, CD to the square of BD; and, by composition, the ratio of four times the rectangle BC, CD together with the square of BD to the square of BD is given. but four times the rectangle BC, CD together with the square of BD is equal to the square of the straight lines BC, CD taken together; therefore the ratio of the square of BC, CD together to the square of BD is given; wherefore the ratio of the straight line BC together with CD to BD is given. and, by composition, the ratio of BC together with CD and DB, that is the ratio of twice BC to BD is given; therefore the ratio of BC to BD is given, as also the ratio of the square of BC to the rectangle CB, BD. but the rectangle CB, BD is given, being the given excess of the squares of BC, BA; therefore the square of BC, and the straight line BC is given, and the ratio of

BC to BD, as also of BD to BA has been shewn to be given; therefore the ratio of BC to BA is given; and BC is given, wherefore BA is given.

The preceding Demonstration is the Analysis of this Problem, viz.

A parallelogram AC which has a given angle ABC being given in magnitude, and the excess of the square of BC one of its sides above the square of the other BA being given; to find the sides. and

The Composition is as follows,

Let EFG be the given angle to which the angle ABC is required to be equal, and from any point E in EE draw EG perpendicular to FG; let the rectangle EG, GH be the given space to which the parallelogram AC is to be made equal; and the rectangle HG, GL be the given excess of the squares of BC, BA.

Take, in the straight line GE, GK equal to FE, and make GM double of GK; join ML, and in GL produced take LN equal to LM. bisect GN in O, and between GH, GO find a mean proportional BC. as OG to GL, so make CB to BD; and make the angle CBA equal to GFE, and as LG to GK, so make DB to BA; and complete the parallelogram AC. AC is equal to the rectangle EG, GH, and the excess of the squares of CB, BA is equal to the rectangle HG, GL.

Because as CB to BD, so is OG to GL, the square of CB is to the rectangle CB, BD, as the rectangle HG, GO to the rectangle HG, GL. and the square of CB is equal to the rectangle HG, GO, because GO, BC, GH are proportionals; therefore the rectangle CB, BD is equal to HG, GL. and because as CB. to BD, so is OG to GL, twice CB is to BD, as twice OG, that is GN, to GL; and, by division, as BC together with CD is to BD, so is NL, that is LM, to LG. therefore the square of BC together with CD is to the square of BD, as the square of ML to the square of LG. but the square of BC and CD together is equal to four times the rectangle BC, CD together with the square of BD; therefore four times the rectangle BC, CD together with the square of BD is to the square of BD, as the square of ML to the square of LG. and, by division, four times the rectangle BC, CD is to the square of BD, as the square of MG to the square of GL; wherefore the rectangle BC, CD is to the square of BD, as (the square of KG the half of MG to the square of GL, that is as) the square of AB to the square of BD, because as LG to GK, so DB was made to BA. therefore the rectangle BC, CD is equal to the square of AB; to each of these add the rectangle CB, BD, and the square of BC becomes equal to the square of AB together with the rectangle CB, BD. therefore this rectangle, that is the given rectangle HG, GL is the excess of the squares of BC, AB. from the point A draw AP perpendicular to BC, and because the angle ABP is equal to the angle EFG, the triangle ABP is equiangular to EFG. and DB was made to BA, as LG to GK, therefore as the rectangle CB, BD to CB, BA, so is the rectangle HG, GL to HG, GK; and as the rectangle CB, BA to

Ap, bc, so is (the straight line BA to AP, and so is FE or GK to EG, and so is) the rectangle HG, GK to HG, GE; therefore, ex aequali, as the rectangle CB, BD to AP, BC, so is the rectangle HG, GL to EG, GH. and the rectangle CB, BD is equal to HG, GL, therefore the rectangle AP, BC, that is the parallelogram AC is equal to the given rectangle EG, GH.

PROP. LXXXVIII.

N.

IF two straight lines contain a parallelogram given in magnitude, in a given angle; if the sum of the squares of its sides be given, the sides shall each of them be given.

Let the two straight lines AB, BC contain the parallelogram ABCD given in magnitude in the given angle ABC, and let the sum of the squares of AB, BC be given; AB, BC are each of them given.

First, let ABC be a right angle; and because twice the rectangle contained by two equal straight lines is equal to both their squares; but if two straight lines are unequal, twice the rectangle contained by them is less than the sum of their squares, as is evident from the 7. Prop. B. 2. Elem. therefore twice the given space, to which space the Rectangle of which the sides are to be found, is equal, must not be greater than the given sum of the squares of the sides. and if twice that space be equal to the given sum of the squares, the sides of the rectangle must necessarily be equal to one another. therefore in this case describe a square ABCD equal to the given rectangle, and its sides AB, BC are those which were to be found. for the rectangle AC is equal to the given space, and the sum of the squares of its sides AB, BC is equal to twice the rectangle AC, that is, by the hypothesis, to the given space to which the sum of the squares was required to be equal.

But if twice the given rectangle be not equal to the given sum of the squares of the sides, it must be less than it, as has been shewn. Let ABCD be the rectangle, join AC and draw BE perpendicular to it, and complete the rectangle AEBF, and describe the circle ABC about the triangle ABC; AC is its diameter. and because the triangle ABC is similar to AEB, as AC to CB, so is AB to BE; therefore the rectangle AC, BE is equal to AB, BC; and the rectangle AB, BC is given, wherefore AC, BE is given. and because the sum of the squares of AB, BC is given, the square of AC which is equal to that sum is given; and AC itself is therefore given in magnitude. let AC be likewise given in position, and the point A; therefore AF is given in position. and the rectangle AC, BE is given, as has been shewn, and AC is given, wherefore BE is given in magnitude, as also AF which is equal to it; and AF is also given in position, and the point A is given; wherefore the point is given, and the straight line FB in position. and the circumference ABC is given in position, wherefore the point B is given. and the points A, C are given; therefore the straight lines AB, BC are given in position and magnitude.

The sides AB, BC of the rectangle may be found thus; let the rectangle GH, GK be the given space to which the rectangle AB, BC is equal; and let GH, GL be the given rectangle to which the sum of the squares of AB, BC is equal. find a square equal to the rectangle GH, GL, and let its side AC be given in position; upon AC as a diameter describe the semicircle ABC, and as AC to GH, so make GK to AF, and from the point A place AF at right angles to AC. therefore the rectangle CA, AF is equal to GH, GK; and, by the hypothesis, twice the rectangle GH, GK is less

than GH, GL, that is than the square of AC; wherefore twice the rectangle CA, AF is less than the square of AC, and the rectangle CA, AF itself less than half the square of AC, that is than the rectangle contained by the diameter AC and its half; wherefore AF is less than the semidiameter of the circle, and consequently the straight line drawn through the point F parallel to AC must meet the circumference in two points. let B be either of them, and join AB, BC and complete the rectangle ABCD; ABCD is the rectangle which was to be found. draw BE perpendicular to AC; therefore BE is equal to AF, and because the angle ABC in a semicircle is a right angle, the rectangle AB, BC is equal to AC, BE, that is to the rectangle CA, AF which is equal to the given rectangle GH, GK. and the squares of AB, BC are together equal to the square of AC, that is to the given rectangle GH, GL.

But if the given angle ABC of the parallelogram AC be not a right angle, in this case because ABC is a given angle, the ratio of the rectangle contained by the sides AB, BC to the parallelogram AC is given; and AC is given, therefore the rectangle AB, BC is given. and the sum of the squares of AB, BC is given; therefore the sides AB, BC are given by the preceding case.

The sides AB, BC and the parallelogram AC may be found thus. let EFG be the given angle of the parallelogram, and from any point E in FE draw EG perpendicular to FG. and let the rectangle EG, FH be the given space to which the parallelogram is to be made equal, and let EF, FK be the given rectangle to which the sum of the squares of the sides is to be equal. and, by the preceding case, find the sides of a rectangle which is equal to the given rectangle EF, FH, and the squares of the sides of which are together equal to the given rectangle EF, FK. therefore, as was shewn in that case, twice the rectangle EF, FH must not be greater than the rectangle EF, FK; let it be so, and let AB, BC be the sides of the rectangle joined in the angle ABC equal to the given angle EFG; and complete the parallelogram ABCD, which will be that which was to be found. draw AL perpendicular to BC, and because the angle ABL is equal to EFG, the triangle ABL is equiangular to EFG. and the parallelogram AC, that is the rectangle AL, BC is to the rectangle AB, BC as (the straight line AL to AB, that is as EG to EF, that is as) the rectangle EG, FH to EF, FH. and, by the construction, the rectangle AB, BC is equal to EF, FH, therefore the rectangle AL, BC, or, its equal, the parallelogram AC is equal to the given rectangle EG, FH. and the squares of AB, BC are together equal, by construction, to the given rectangle EF, FK.

PROP. LXXXIX.

86. IF two straight lines contain a given parallelogram in a given angle, and if the excess of the square of one of them above a given space has a given ratio to the square of the other; each of the straight lines shall be given.

Let the two straight lines AB, BC contain the given parallelogram AC in the given angle ABC, and let the excess of the square of BC above a given space have a given ratio to the square of AB; each of the straight lines AB, BC is given.

Because the excess of the square of BC above a given space has a given ratio to the square of BA, let the rectangle CB, BD be the given space; take this from the square of BC, the remainder, to wit, the rectangle BC, CD has a given ratio to the square of BA. draw AE perpendicular to BC, and let the square of BF be equal to the rectangle BC, CD. then because the angle ABC, as also BEA is given, the triangle ABE is given in species, and the ratio of AE to AB given. and because the ratio of the rectangle BC, CD, that is of the square of BF to the square of BA is given, the ratio of the straight line BF to BA is given. and the ratio of AE to AB is given, wherefore the ratio of AE to BF is given, as also the ratio of the rectangle AE, BC, that is of the parallelogram AC to the rectangle FB, BC; and AC is given, wherefore the rectangle FB, BC is given. and the excess of the square of BC above the square of BF, that is above the rectangle BC, CD is given, for it is equal to the given rectangle CB, BD. therefore because the rectangle contained by the straight lines FB, BC is given, and also the excess of the square of BC above the square of BF; FB, BC are each of them given. and the ratio of FB to BA is given; therefore AB, BC are given.

The Composition is as follows.

Let GHK be the given angle to which the angle of the parallelogram is to be made equal, and from any point G in HG draw GK perpendicular to HK; let GK, HL be the rectangle to which the parallelogram is to be made equal, and let LH, HM be the rectangle equal to the given space which is to be taken from the square of one of the sides; and let the ratio of the remainder to the square of the other side be the same with the ratio of the square of the given straight line NH to the square of the given straight line HG.

By help of the 87. Dat. find two straight lines BC, BF which contain a rectangle equal to the given rectangle NH, HL, and such that the excess of the square of BC above the square of BF be equal to the given rectangle LH, HM; and join CB, BF in the angle FBC equal to the given angle GHK. and as NH to HG, so make FB to BA, and complete the parallelogram AC, and draw AE perpendicular to BC. then AC is equal to the rectangle GK, HL; and if from the square of BC, the given rectangle LH, HM be taken, the remainder shall have to the square of BA the same ratio which the square of NH has to the square of HG.

Because, by the construction, the square of BC is equal to the square of BF together with the rectangle LH, HM; if from the square of BC there be taken the rectangle LH, HM, there remains the square of BF which has to the square of BA the same ratio which the square of NH has to the square of HG, because as NH to HG, so FB was made to BA; but as HG to GK, so is BA to AE, because the triangle GHK is equiangular to ABE; therefore, ex aequali, as NH to GK, so is FB to AE. wherefore the rectangle NH, HL is to the rectangle GK, HL, as the rectangle FB, BC to AE, BC. but, by the construction, the rectangle NH, HL is equal to FB, BC; therefore the rectangle GK, HL is equal to the rectangle AE, BC, that is to the parallelogram AC.

The Analysis of this Problem might have been made as in the 86. Prop. in the Greek, and the composition of it may be made as that which is in Pro. of this Edition.

PROP. XC.

O.

IF two straight lines contain a given parallelogram in a given angle, and if the square of one of them be given together with the space which has a given ratio to the square of the other; each of the straight lines shall be given.

Let the two straight lines AB, BC contain the given parallelogram AC in the given angle ABC, and let the square of BC be given together with the space which has a given ratio to the square of AB; AB, BC are each of them given.

Let the square of BD be the space which has the given ratio to the square of AB; therefore, by the hypothesis, the square of BC together with the square of BD is given. from the point A draw AE perpendicular to BC, and because the angles ABE, BEA are given, the triangle ABE is given in species; therefore the ratio of BA to AE is given. and because the ratio of the square of BD to the square of BA is given, the ratio of the straight line BD to BA is given; and the ratio of BA to AE is given, therefore the ratio of AE to BD is given, as also the ratio of the rectangle AE, BC, that is of the parallelogram AC to the rectangle DB, BC. and AC is given, therefore the rectangle DB, BC is given; and the square of BC together with the square of BD is given, therefore because the rectangle contained by the two straight lines DB, BC is given, and the sum of their squares is given; the straight lines DB, BC are each of them given. and the ratio of DB to BA is given; therefore AB, BC are given.

The Composition is as follows.

Let FGH be the given angle to which the angle of the parallelogram is to be made equal, and from any point F in GF draw FH perpendicular to GH; and let the rectangle FH, GK be that to which the parallelogram is to be made equal; and let the rectangle KG, GL be the space to which the square of one of the sides of the parallelogram together with the space which has a given ratio to the square of the other side, is to be made equal; and let this given ratio be the same which the square of the given straight line MG has to the square of GF.

By the 83. Dat. find two straight lines DB, BC which contain a rectangle equal to the given rectangle MG, GK, and such that the sum of their squares is equal to the given rectangle KG, GL therefore, by the determination of the Problem in that Proposition, twice the rectangle MG, GK must not be greater than the rectangle KG, GL. let it be so, and join the straight lines DB, BC in the angle DBC equal to the given angle FGH. and as MG to

GF, so make DB to BA, and complete the parallelogram AC. AC is equal to the rectangle FH, GK; and the square of BC together with the square of BD which by the construction has to the square of BA the given ratio which the square of MG has to the square of GF, is equal, by the construction, to the given rectangle KG, GL. Draw AE perpendicular to BC.

Because as DB to BA, so is MG to GF; and as BA to AE, so GF to FH; ex aequali, as DB to AE, so is MG to FH. therefore as the rectangle DB, BC to AE, BC, so is the rectangle MG, GK to FH, GK. and the rectangle DB, BC is equal to the rectangle MG, GK; therefore the rectangle AE, BC, that is the parallelogram AC, is equal to the rectangle FH, GK.

PROP. XCI.

88. IF a straight line drawn within a circle given in magnitude cuts off a segment which contains a given angle; the straight line is given in magnitude.

In the circle ABC given in magnitude, let the straight line AC be drawn cutting off the segment AEC which contains the given angle AEC; the straight line AC is given in magnitude.

Take D the center of the circle, join AD and produce it to

E, and join EC. the angle ACE being a right angle is given; and the angle AEC is given; therefore the triangle ACE is given in species, and the ratio of EA to

AC is therefore given. and EA is given in magnitude, because the circle is given in magnitude; AC is therefore given in magnitude.

PROP. XCII.

89. IF a straight line given in magnitude be drawn within a circle given in magnitude; it shall cut off a segment containing a given angle.

Let the straight line AC given in magnitude be drawn within the circle ABC given in magnitude; it shall cut off a segment containing a given angle.

Take D the center of the circle, join AD and produce it to E, and join EC. and because each of the straight lines EA, AC is given, their ratio is given; and the angle ACE is a right angle, therefore the triangle ACE is given in species, and consequently the angle AEC is given.

PROP. XCIII. 90.

IF from any point in the circumference of a circle given in position two straight lines be drawn meeting the circumference and containing a given angle; if the point in which one of them meets the circumference again be given, the point in which the other meets it is also given.

From any point A in the circumference of a circle ABC given in position, let AB, AC be drawn to the circumference making the given angle BAC; if the point B be given, the point C is also given.

Take D the center of the circle, and join BD, DC. and because each of the points B, D is given, BD is given in position. and because the angle BAC is given, the angle BDC is given. therefore because the straight line DC is drawn to the given point D in the straight line BD given in position in the given angle BDC, DC is given in position. and the circumference ABC is given in position, therefore the point C is given.

PROP. XCIV.

91. IF from a given point a straight line be drawn touching a circle given in position; the straight line is given in position and magnitude.

Let the straight line AB be drawn from the given point A touching the circle BC given in position; AB is given in position and magnitude.

Take D the center of the circle, and join DA, DB. because each of the points D, A is given, the straight line AD is given in position and magnitude. and

DBA is a right angle, wherefore DA is a diameter of the circle DBA described about the triangle DBA; and that circle is therefore given in position. and the circle BC is given in position, therefore the point B is given. the point A is also given; therefore the straight line AB is given in position and magnitude.

PROP. XCV.

92. IF a straight line be drawn from a given point without a circle given in position; the rectangle contained by the segments betwixt the point and the circumference of the circle is given.

Let the straight line ABC be drawn from the given point A without the circle BCD given in position, cutting it in B, C; the rectangle BA, AC is given.

From the point A draw AD touching the circle; therefore AD is given in position and magnitude. and because AD is given, the square of AD is given which is equal to the rectangle BA, AC. therefore the rectangle BA, AC is given.

PROP. XCVI.

93. IF a straight line be drawn thro' a given point within a circle given in position, the rectangle contained by the segments betwixt the point and the circumference of the circle is given.

Let the straight line BAC be drawn thro' the given point A within the circle BCE given in position; the rectangle BA, AC is given.

Take D the center of the circle, join AD and produce it to the points E, F. because the points A, D are given, the straight line AD is given in position; and the circle BEC is given in position; therefore the points E, F are given. and the point A is given, therefore EA, AF are each of them given; and the rectangle EA, AF is therefore given; and it is equal to the rectangle BA, AC which consequently is given.

PROP. XCVII.

94. IF a straight line be drawn within a circle given in magnitude cutting off a segment containing a given angle; if the angle in the segment be bisected by a straight line produced till it meets the circumference, the straight lines which contain the given angle shall both of them together have a given ratio to the straight line which bisects the angle. and the rectangle contained by both these lines together which contain the given angle, and the part of the bisecting line cut off below the base of the segment, shall be given.

Let the straight line BC be drawn within the circle ABC given in magnitude cutting off a segment containing the given angle BAC. and let the angle BAC be bisected by the straight line AD; BA together with AC has a given ratio to AD; and the rectangle contained by BA and AC together, and the straight line ED cut off from AD below BC the base of the segment, is given.

Join BD; and because BC is drawn within the circle ABC given in magnitude cutting off the segment BAC containing the given angle BAC; BC is given in magnitude. by the same reason BD is given; therefore the ratio of BC to BD is given. and because the angle BAC is bisected by AD, as BA to AC, so is BE to EC; and, by permutation, as AB to BE, so is AC to CE; wherefore as BA and AC together to BC, so is AC to CE. and because the angle BAE is equal to EAC, and the angle ACE to ADB; the triangle ACE is equiangular to the triangle ADB; therefore as AC to CE, so is AD to DB. but as AC to CE, so is BA together with AC to BC; as therefore BA and AC to BC, so is AD to DB; and, by permutation, as BA and AC to AD, so is BC to BD. and the ratio of BC to BD is given, therefore the ratio of BA together with AC to AD is given.

Also the rectangle contained by BA and AC together, and DE is given.

Because the triangle BDE is equiangular to the triangle ACE, as BD to DE, so is AC to CE; and as AC to CE, so is BA and AC to BC; therefore as BA and AC to BC, so is BD to DE. wherefore the rectangle contained by BA and AC together, and DE is equal to the rectangle CB, BD. but CB, BD is given; therefore the rectangle contained by BA and AC together, and DE is given.

Otherwise.

Produce CA and make AF equal to AB, and join BF. and because the angle BAC is double of each of the angles BFA, BAD, the angle BFC is equal to BAD; and the angle BCA is equal to BDA, therefore the triangle FCB is equiangular to ADB. as therefore FC to CB, so is AD to DB, and, by permutation, as FC, that is BA and AC together to AD, so is CB to BD. and the ratio of CB to BD is given, therefore the ratio of BA and AC to AD is given.

And because the angle BFC is equal to the angle DAC, that is to the angle DBC, and the angle ACB equal to the angle ADB; the triangle FCB is equiangular to BDE, as therefore FC to CB, so is BD to DE; therefore the rectangle contained by FC, that is BA and AC together, and DE is equal to the rectangle CB, BD which is given, and therefore the rectangle contained by BA, AC together, and DE is given.

PROP. XCVIII.

P.

IF a straight line be drawn within a circle given in magnitude cutting off a segment containing a given angle; it the angle adjacent to the angle in the segment be bisected by a straight line produced till it meet the circumference again and the base of the segment; the excess of the straight lines which contain the given angle shall have a given ratio to the segment of the bisecting line which is within the circle; and the rectangle contained by the same excess and the segment of the bisecting line betwixt the base produced and the point where it again meets the circumference, shall be given.

Let the straight line BC be drawn within the circle ABC given in magnitude cutting off a segment containing the given angle BAC, and let the angle CAF adjacent to BAC be bisected by the straight line DAE meeting the circumference again in D, and BC the base of the segment produced in E; the excess of BA, AC has a given ratio to AD; and the rectangle which is contained by the same excess and the straight line ED, is given.

Join BD, and thro' B draw BG parallel to DE meeting AC produced in G. and because BC cuts off from the circle ABC given in magnitude the segment BAC containing a given angle, BC is therefore given in magnitude. by the same reason BD is given, because the angle BAD is equal to the given angle EAF; therefore the ratio of BC to BD is given. and because the angle CAE is equal to EAF, of which CAE is equal to the alternate angle AGB, and EAF to the interior and opposite angle ABG; therefore the angle AGB is equal to ABG, and the straight line AB equal to AG; so that GC is the excess of BA, AC. and because the angle BGC is equal to GAE, that is to EAF, or the angle BAD; and that the angle BCG is equal to the opposite interior angle BDA of the quadrilateral BCAD in the circle; therefore the triangle BGC is equiangular to BDA therefore as CC to CB, so is AD to DB, and, by permutation, as GC, which is the excess of BA, AC to AD, so is CB to BD. and the ratio of CB to BD is given; therefore the ratio of the excess of BA, AC to AD is given.

And because the angle GBC is equal to the alternate angle DEB, and the angle BCG equal to BDE; the triangle BCG is equiangular to BDE. therefore as GC to CB, so is BD to DE, and consequently the rectangle GC, DE is equal to the rectangle CB, BD which is given, because its sides CB, BD are given. therefore the rectangle contained by the excess of BA, AC and the straight line DE is given.

PROP. XCIX.

95. IF from a given point in the diameter of a circle given in position, or in the diameter produced, a straight line be drawn to any point in the circumference, and from that point a straight line be drawn at right angles to the first, and from the point in which this meets the circumference again, a straight line be drawn parallel to the first; the point in which this parallel meets the diameter is given; and the rectangle contained by the two parallels is given.

In BC the diameter of the circle ABC given in position, or in BC produced, let the given point D be taken, and from D let a straight line DA be drawn to any point A in the circumference, and let AE be drawn at right angles to DA, and from the point E where it meets the circumference again let EF be drawn parallel to DA meeting BC in F; the point F is given, as also the rectangle AD, EF.

Produce EF to the circumference in G, and join AG. because GEA is a right angle, the straight line AG is the diameter of the circle ABC; and BC is also a diameter of it; therefore the point H where they meet is the center of the circle, and consequently H is given. and the point D is given, wherefore DH is given in magnitude. and because AD is parallel to FG, and GH equal to HA; DH is equal to HF, and AD equal to GF. and DH is

given, therefore HF is given in magnitude; and it is also given in position, and the point H is given, therefore the point F is given.

And because the straight line EFG is drawn from a given point F without or within the circle ABC given in position, therefore the rectangle EF, FG is given. and GF is equal to AD, wherefore the rectangle AD, EF is given.

PROP. C.

Q

IF from a given point in a straight line given in position, a straight line be drawn to any point in the circumference of a circle given in position; and from this point a straight line be drawn making with the first an angle equal to the difference of a right angle and the angle contained by the straight line given in position, and the straight line which joins the given point and the center of the circle; and from the point in which the second line meets the circumference again, a third straight line be drawn making with the second an angle equal to that which the first makes with the second. the point in which this third line meets the straight line given in position is given; as also the rectangle contained by the first straight line and the segment of the third betwixt the circumference and the straight line given in position, is given.

Let the straight line CD be drawn from the given point C in the straight line AB given in position, to the circumference of the circle DEF given in position of which G is the center; join CG, and from the point D let DF be drawn making the angle CDF equal to the difference of a right angle and the angle BCG, and from the point F let FE be drawn making the angle DFE equal to CDF, meeting AB in H. the point H is given; as also the rectangle CD, FH.

Let CD, FH meet one another in the point K from which draw KL perpendicular to DF; and let DC meet the circumference again in M, and let FH meet the same in E, and join MG, GF, CH.

Because the angles MDF, DFE are equal to one another, the circumferences MF, DE are equal; and adding or taking away the common part ME, the circumference DM is equal to EF; therefore the straight line DM is equal to the straight line EF, and the angle GMD to the angle GFE; and the angles GMC, GFH are equal to one another, because they are either the same with the angles GMD, GFE, or adjacent to them. and because the angles KDL, LKD are together equal to a right angle, that is, by the hypothesis, to the angles KDL, GCB; the angle GCB or GCH is equal to the angle (LKD, that is to the angle) LKF or GKH. therefore the points C, K, H, G are in the circumference of a circle; and the angle GCK is therefore equal to the angle GHF; and the angle CMC is equal to GFH, and the straight line GM to GF; therefore CG is equal to GH, and CM to HF. and because CG is equal to GH, the angle GCH is equal to GHC; but the angle GCH is given, therefore GHC is given; and consequently the angle CGH is given. and CG is given in position, and the point G; therefore GH is given in position; and CB is also given in position, wherefore the point H is given.

And because HF is equal to CM, the rectangle DC, FH is equal to DC, CM. but DC, CM is given, because the point C is given; therefore the rectangle DC, FH is given.

FINIS

NOTES.

DEFINITION II. THIS IS MADE more explicit than in the Greek text, to prevent a mistake which the Author of the second Demonstration of the 24th Proposition in the Greek Edition has fallen into, of thinking that a ratio is given to which another ratio is shewn to be equal, tho' this other be not exhibited in given magnitude. See the Notes on that Proposition which is the 13 th in this Edition. besides by this Definition, as it is now given, some Propositions are demonstrated, which in the Greek are not so well done by help of Pro.

DEF. IV.

In the Greek text Def. 4. is thus "Points, lines, spaces and angles are said to be given in position which have always the same situation." but this is imperfect and useless, because there are innumerable cases in which things may be given according to this Definition, and yet their position cannot be found. for instance, let the triangle ABC be given in position, and let it be proposed to draw a straight line BD from the angle at B to the opposite side AC which shall cut off the angle DBC which shall be the seventh part of the angle ABC. suppose this is done, therefore the straight line BD is invariable in its position, that is, has always the same situation; for any other straight line drawn from the point B on either side of BD cuts off an angle greater or lesser than the seventh part of the angle ABC; therefore, according to this Definition, the straight line BD is given in position, as also the point D in which it meets the straight line AC which is given in position. but from the things here given, neither the straight line BD nor the point D can be found by the help of Euclid's Elements only, by which every thing in his Data is supposed may be found. this Definition is therefore of no use. we have amended it by adding "and which are either actually exhibited or can be found;" for nothing is to be reckoned given, which cannot be found, or is not actually exhibited.

The Definition of an angle given by position is taken out of the 4th, and given more distinctly by itself in the Definition marked A.

DEF. XI. XII. XIII. XIV. XV.

The 11th and 12th are omitted because they cannot be given in English so as to have any tolerable sense, and therefore wherever the terms defined occur, the words which express their meaning are made use of in their place.

The 13. 14. 15. are omitted as being of no use.

It is to be observed in general of the Data in this book, that they are to be understood to be given Geometrically, not always Arithmetically, that is, they cannot always be exhibited in numbers; for instance, if the side of a square be given, the ratio of it to its diameter is given geometrically, but not in numbers; and the diameter is given, but tho' the number of any equal parts in the side be given, for example to, the number of them in the diameter cannot be given. and the like holds in many other cases.

PROPOSITION I. In this it is shewn that A is to B, as C to D, from this that A is to C, as B to D, and then by permutation; but it follows directly, without these two steps, from 7. 5.

PROP. II.

The limitation added at the end of this Proposition between the inverted commas is quite necessary, because without it the Proposition cannot always be demonstrated. for the Author having said "because A is given, a magnitude equal to it can be found, let this be C; and because the ratio of A to B is given, a ratio which is the same to it can be found" adds, "let it be found, and let it be the ratio of C to ." Now from the second Definition nothing more follows than that some ratio, suppose the ratio of E to Z, can be found, which is the same with the ratio of A to B; and when the Author supposes that the ratio of C to , which is

also the same with the ratio of A to B, can be found, he necessarily supposes that to the three magnitudes E, Z, C a fourth proportional may be found; but this cannot always be done by the Elements of Euclid; from which it is plain Euclid must have understood the Proposition under the limitation which is now added to his text. An example will make this clear; let A be a given angle, and B another angle to which A has a given ratio, for instance, the ratio of the given straight line E to the given one Z, then, having found an angle C equal to A, how can the angle be found to which C has the same ratio that E has to Z? certainly no way, until it be shewn how to find an angle to which a given angle has a given ratio, which cannot be done by Euclid's Elements, nor probably by any Geometry known in his time. Therefore in all the Propositions of this book which depend upon this second, the above-mentioned limitation must be understood, tho' it be not explicitly mentioned.

PROP. V.

The order of the Propositions in the Greek text between Pro. and Pro. is now changed into another which is more natural, by placing those which are more simple before those which are more complex; and by placing together those which are of the same kind, some of which were mixed among others of a different kind. thus Pro. in the Greek is now made the 5. and those which were the 22. and 23. are made the 11. and 12. as they are more simple than the Propositions concerning magnitudes the excess of one of which above a given magnitude has a given ratio to the other, after which these two were placed; and the 24. in the Greek text is, for the same reason, made the 13.

PROP. VI. VII.

These are universally true, tho' in the Greek text they are demonstrated by Pro. which has a limitation; they are therefore now shewn without it.

PROP. XII.

In the 23. Prop. in the Greek text, which here is the 12. the words "" are wrong translated by Claud. Hardy in his Edition of Euclid's Data printed at Paris Ann., 1625, which was the first Edition of the Greek text; and Dr. Gregory follows him in translating them by the words "etsi non easdem, "as if the Greek had been as in Pro. of the Greek text. Euclid's meaning is that the ratios mentioned in the Proposition must not be the same; for if they were, the Proposition would not be true. whatever ratio the whole has to the whole, if the ratios of the parts of the first to the parts of the other be the same with this ratio, one part of the first may be double, triple, &c. of the other part of it, or have any other ratio to it, and consequently cannot have a given ratio to it. wherefore these words must be rendered by "non autem easdem, "but not the same ratios, as Zambertus has translated them in his Edition.

PROP. XIII.

Some very ignorant Editor has given a second Demonstration of this Proposition in the Greek text, which has been as ignorantly kept in it by Claud. Hardy and Dr. Gregory, and has been retained in the translations of Zambertus and others; Carolus Renaldinus gives it only. the author or it has thought that a ratio was given if another ratio could be shewn to be the same to it, tho' this last ratio be not found. but this is altogether absurd, because from it would be deduced that the ratio of the sides of any two squares is given, and the ratio of the diameters of any two circles, &c. and it is to be observed that the moderns frequently take given ratios, and ratios that are always the same for one and the same thing, and Sir Isaac Newton has fallen into this mistake in the 17th Lemma of his Principia, Ed. 1713. and in other places. but this should be carefully avoided, as it may lead into other errors.

PROP. XIV. XV.

Euclid in this book has several Propositions concerning magnitudes, the excess of one of which above a given magnitude has a given ratio to the other; but he has given none concerning magnitudes whereof one together with a given magnitude has a given ratio to the other; tho' these last occur as frequently in the solution or Problems as the first. the reason of which is, that the last may be all demonstrated by help of the first; for if a magnitude together with a given magnitude has a given ratio to another magnitude; the excess of this other above a given magnitude shall have a given ratio to the first, and on the contrary; as we have demonstrated in Pro. and for a like reason Pro. has been added to the Data. one example will make the thing clear; suppose it were to be demonstrated, That if a magnitude A together with a given magnitude has a given ratio to another magnitude B, that the two magnitudes A and B, together with a given magnitude have a given ratio to that other magnitude B; which is the same Proposition with respect to the last kind of magnitudes above-mentioned, that the first part of Pro. in this Edition is in respect of the first kind. this is shewn thus; from the hypothesis, and by the first part of Pro. the excess of B above a given magnitude has unto A a given ratio; and therefore, by the first part of Pro. the excess of B above a given magnitude has unto B and A together a given ratio; and by the second part of Pro. A and B together with a given magnitude has unto B a given ratio; which is the thing that was to be demonstrated. in like manner the other Propositions concerning the last kind of magnitudes may be shewn.

PROP. XVI. XVII.

In the third part of Pro. in the Greek text, which is the 16. in this Edition, after the ratio of EC to CB has been shewn to be given; from this, by inversion and conversion, the ratio of BC to BE is demonstrated to be given; but, without these two steps, the conclusion should have been made only by citing the 6. Proposition. and in like manner, in the first part of Pro. in the Greek, which in this Edition is the 17. from the ratio of DB to BC being given, the ratio of DC to DB is shewn to be given, by inversion and Composition, instead of citing Pro, and the same fault occurs in the second part of the same Pro.

PROP. XXI. XXII.

These are now added, as being wanting to complete the subject treated of in the four preceding Propositions.

PROP. XXIII.

This which is Pro. in the Greek text, was separated from Pro. 15. 16. in that text, after which it should have been immediately placed, as being of the same kind. it is now put into its proper place. but Pro. in the Greek is left out, as being the same with Pro. in that text, which is here Pro.

PROP. XXIV.

This, which is Pro. in the Greek, is now put into its proper place, having been disjoined from the three following it in this Edition, which are of the same kind.

PROP. XXVIII.

This which in the Greek text is Pro. and several of the following Propositions, are there deduced from Def. 4. which is not sufficient, as has been mentioned in the Note on that Definition; they are therefore now shewn more explicitly.

PROP. XXXIV. XXXVI.

Each of these has a Determination, which is now added, which occasions a change in their Demonstrations.

PROP. XXXVII. XXXIX. XL. XLI.

The 35. and 36. Propositions in the Greek text are joined into one, which makes the 39. in this Edition, because the same Enuntiation and

Demonstration serves both. and for the same reason Pro. 38. in the Greek are joined into one which here is the 40.

Pro. is added to the Data, as it frequently occurs in the solution of Problems. and Pro. is added to complete the rest.

PROP. XLII.

This is Pro. in the Greek text, where the whole construction of Pro. of Book 1. of the Elements is put without need into the Demonstration, but is now only cited.

PROP. XLV.

This is Pro. in the Greek, where the three straight lines made use of in the construction are said, but not shewn, to be such that any two of them is greater than the third, which is now done.

PROP. XLVII.

This is Pro. in the Greek text, but the Demonstration of it is changed into another wherein the several cases of it are shewn, which, tho' necessary, is not done in the Greek.

PROP. XLVIII.

There are two cases in this Proposition, arising from the two cases of the 3d part of Pro. on which the 48. depends. and in the Composition these two cases are explicitly given.

PROP. LII.

The Construction and Demonstration of this which is Pro. in the Greek, are made something shorter than in that text.

PROP. LIII.

Pro. in the Greek text is omitted, being only a case of Pro. in that text, which is Pro. in this Edition.

PROP. LVIII.

This is not in the Greek text, but its Demonstration is contained in that of the first part of Pro. in that text; which Proposition is concerning figures that are given in species; this 58. is true of similar figures, tho' they be not given in species, and as it frequently occurs, it was necessary to add it.

PROP. LIX. LXI.

This is the 54. in the Greek; and the 77. in the Greek, being the very same with it, is left out. and a shorter Demonstration is given of Pro.

PROP. LXII.

This which is most frequently useful is not in the Greek, and is necessary to Pro. 88. in this Edition, as also, tho' not mentioned, to Pro. 87. in the former Editions. Pro. in the Greek text is made a Corollary to it.

PROP. LXIV.

This contains both Pro. and 73. in the Greek text; the first case of the 74. is a repetition of Pro. from which it is separated in that text by many Propositions; and as there is no order in these Propositions, as they stand in the Greek, they are now put into the order which seemed most convenient and natural.

The Demonstration of the first part of Pro. in the Greek is grossly vitiated. Dr. Gregory says that the sentences he has inclosed betwixt two stars are superfluous and ought to be cancelled; but he has not observed that what follows them is absurd, being to prove that the ratio [see his figure] of AT to TK is given, which by the Hypothesis at the beginning of the Proposition is expressly given; so that the whole of this part was to be altered, which is done in this Pro.

PROP. LXVII. LXVIII.

Pro. in the Greek text is divided into these two, for the sake of distinctness; and the Demonstration of the 67. is rendered shorter than that of the first part of Pro. in the Greek by means of Pro. of Book 6. of the Elements.

PROP. LXX.

This is Pro. in the Greek text; Pro. in that text is only a particular case of it, and is therefore omitted.

Dr. Gregory in the Demonstration of Pro. cites the 49. Prop. Dat. to prove that the ratio of the figure AEB to the parallelogram AH is given, whereas this was shewn a few lines before; and besides the 49. Prop. is not applicable to these two figures, because AH is not given in species, but is, by the step for which the citation is brought, proved to be given in species.

PROP. LXXIII.

Pro. in the Greek text is neither well enuntiated nor demonstrated. the 73. which in this Edition is put in place of it, is really the same, as will appear by considering [see Dr. Gregory's Edition] that A, B, D, E in the Greek text are four proportionals, and that the Proposition is to shew that , which has a given ratio to E, is to D, as B is to the straight line to which A has a given ratio; or, by inversion, that is to , as the straight line to which A has a given ratio is to B; that is, if the proportionals be placed in this order, viz. D, E, A, B, that the first P is to to which the second E has a given ratio, as the straight line to which the third A has a given ratio is to the fourth B; which is the Enuntiation of this 73. and was thus changed that it might be made like to that at Pro. in this Edition, which is the 82. in the Greek text. and the Demonstration of Pro. is the same with that of Pro. only making use of Pro. instead of Pro. of Book 5. of the Elements.

PROP. LXXVII.

This is put in place of Pro. in the Greek text which is not a Datum, but a Theorem premised as a Lemma to Pro. in that text, and Pro. is made Cor. 1. to Pro. in this Edition. Cl. Hardy in his Edition of the Data takes notice, that, in Pro. of the Greek text, the parallel KL in the figure of Pro. in this Edition must meet the circumference, but does not demonstrate it, which is done here at the end of Cor. 3. of Pro. in the construction for finding a triangle similar to ABC.

PROP. LXXVIII.

The Demonstration of this which is Pro. in the Greek is rendered a good deal shorter by help of Pro.

PROP. LXXIX. LXXX. LXXXI.

These are added to Euclid's Data, as Propositions which are often useful in the solution of Problems.

PROP. LXXXII.

This which is Pro. in the Greek text is placed before the 83. and 84. which in the Greek are the 58. and 59. because the Demonstration of these two in this Edition is deduced from that of Pro. from which they naturally follow.

PROP. LXXXVIII. XC.

Dr. Gregory in his preface to Euclid's Works which he published at Oxford in 1703, after having told that he had supplied the defects of the Greek text of the Data in innumerable places from several Manuscripts, and corrected Cl. Hardy's translation by Mr. Bernard's, adds, that the 86. Theorem " or Proposition, " seemed to be remarkably vitiated, but which could not be restored by help of the Manuscripts; then he gives three different translations of it in Latin, according to which he thinks it may be read; the two first have no distinct meaning, and the third which he says is the best, tho' it contains a true Proposition which is the 90. in this Edition, has no connexion in the least with the Greek text. and it is strange that Dr. Gregory did not observe, that if Pro. was changed into this, the Demonstration of the 86. must be cancelled, and another put in its place. but, the truth is, both the Enuntiation and the Demonstration of Pro. are quite entire and right, only Pro. which is more simple, ought to have been placed before it; and the deficiency which the Doctor justly observes to be in this part of Euclid's Data, and which no doubt is owing to the carelessness and ignorance of the Greek Editors should have been supplied, not by changing Pro. which is both entire and necessary, but by adding the two Propositions which are the 88. and 90. in this Edition.

PROP. XCVIII. C.

These were communicated to me by two excellent Geometers, the first of them by the Right Honourable the Earl Stanhope, and the other by Dr. Matthew Stewart; to which I have added the Demonstrations.

Tho' the order of the Propositions has been in many places changed from that in former Editions, yet this will be of little disadvantage, as the antient Geometers never cite the Data, and the Moderns very rarely.

AS that part of the Composition of a Problem which is its Construction may not be so readily deduced from the Analysis by beginners; for their sake the following Example is given in which the derivation of the several parts of the Construction from the Analysis is particularly shewn, that they may be assisted to do the like in other Problems.

PROBLEM.

Having given the magnitude of a parallelogram, the angle of which ABC is given, and also the excess of the square of its side BC above the square of the side AB; To find its sides and describe it.

The Analysis of this is the same with the Demonstration of the 87. Prop. of the Data. and the Construction that is given of the Problem at the end of that Proposition, is thus derived from the Analysis.

Let EFG be equal to the given angle ABC, and because in the Analysis it is said that the ratio of the rectangle AB, BC to the parallelogram

AC is given by the 62. Prop. Dat. therefore from a point in FE, the perpendicular EG is drawn to FG, as the ratio of FE to EG is the ratio of the rectangle AB, BC to the parallelogram AC by what is shewn at the end of Pro. Next the magnitude of AC is exhibited by making the rectangle EG, GH equal to it, and the given excess of the square of BC above the square of BA, to which excess the rectangle CB, BD is equal, is exhibited by the rectangle HG, GL. then in the Analysis the rectangle AB, BC is said to be given, and this is equal to the rectangle FE, GH, because the rectangle AB, BC is to the parallelogram AC, as (FE to EG, that is as the rectangle) FE, GH to EG, GH; and the parallelogram AC is equal to the rectangle EG, GH, therefore the rectangle AB, BC is equal to FE, GH. and consequently the ratio of the rectangle CB, BD, that is of the rectangle HG, GL, to AB, BC, that is of the straight line DB to BA, is the same with the ratio (of the rectangle GL, GH to FE, GH, that is) of the straight line GL to FE, which ratio of DB to BA is the next thing said to be given in the Analysis. from this it is plain that the square of FE is to the square of GL, as the square of BA which is equal to the rectangle BC, CD is to the square of BD, the ratio of which spaces is the next thing said to be given. and from this it follows that four times the square of FE is to the square of GL, as four times the rectangle BC, CD is to the square of BD; and, by Composition, four times the square of FE together with the square of GL is to the square of GL, as four times the rectangle BC, CD together with the square of BD, is to the square of BD, that is [8. 6.] as the square of the straight lines BC, CD taken together is to the square of BD, which ratio is the next thing said to be given in the Analysis. and because four times the square of FE and the square of GL are to be added together, therefore in the perpendicular EG there is taken KG equal to FE, and MG equal to the double of it, because thereby the squares of MG, GL, that is, joining ML, the square of ML is equal to four times the square of FE and to the square of GL. and because the square of ML is to the square of GL, as the square of the straight line made up of BC and CD is to the square of BD, therefore [22. 6.] ML is to LG, as BC together with CD is to BD, and, by Composition, ML and LG together, that is, producing GL to N, so that ML be equal to LN, the straight line NG is to GL, as twice BC is to BD; and by taking GO equal to the half of NG, GO is to GL, as BC to BD the ratio of which is said to be given in the Analysis. and from this it follows, that the rectangle HG, GO is to HG, GL, as the square of BC is to the rectangle CB, BD which is equal to the rectangle HG, GL, and therefore the square of BC is equal to the rectangle HG, GO, and BC is consequently found by taking a mean proportional betwixt HG and GO, as is said in the Construction. and because it was shewn that GO is to GL, as BC to BD, and that now the three first are found, the fourth BD is found by 12. 6. it was likewise shewn that LG is to FE, or GK, as DB to BA, and the three first are now found, and there by the fourth BA. make the angle ABC equal to EFG, and complete the parallelogram of which the sides are AB, BC, and the construction is finished; the rest of the Composition contains the Demonstration.

AS the Propositions from the 13. to the 28. may be thought by beginners to be less useful than the rest, because they cannot so readily see how they are to be made use of in the solution of Problems; on this account the two following Problems are added, to shew that they are equally useful with the other Propositions, and from which it may easily be judged that many other Problems depend upon these Propositions.

PROBLEM 1.

TO find three straight lines such, that the ratio of the first to the second is given; and if a given straight line be taken from the second, the ratio of the remainder to the third is given; also the rectangle contained by the first and third is given.

Let AB be the first straight line, CD the second, and EF the third. and because the ratio of AB to CD is given, and that if a given straight line be taken from CD, the ratio of the remainder EF is given; therefore the excess of the first AB above a given straight line has a given ratio to the third EF. Let BH be that given straight line, therefore AH the excess of AB above it has a given ratio to EF; and consequently the rectangle BA, AH has a given ratio to the rectangle AB, EF, which last rectangle is given by the Hypothesis; therefore the rectangle BA, AH is given, and BH the excess of its sides is given; wherefore the sides

AB, AH are given. and because the ratios of AB to CD, and of AH to EF are given; CD and EF are given.

The Composition.

Let the given ratio of KL to KM be that which AB is required to have to CD; and let DG be the given straight line which is to be taken from CD, and let the given ratio of KM to KN be that which the remainder must have to EF; also, let the given rectangle NK, KO be that to which the rectangle AB, EF is required to be equal. find the given straight line BH which is to be taken from AB, which is done, as plainly appears from Pro. Dat.

by making as KM to KL, so GD to HB. to the given straight line BH apply a rectangle equal to LK, KO exceeding by a square, and let BA, AH be its sides. then is AB the first of the straight lines required to be found. and by making as LK to KM, so AB to DC, DC will be the second. and lastly, make as KM to KN, so CG to EF, and EF is the third.

For as AB to CD, so is HB to GD, each of these ratios being the same with the ratio of LK to KM; therefore AH is to CG, as (AB to CD, that is, as) LK to KM; and as CG to EF, so is KM to KN; wherefore, ex aequali, as AH to EF, so is LK to KN. and as the rectangle BA, AH to the rectangle BA, EF, so is the rectangle LK, KO to the rectangle KN, KO. and, by the Construction, the rectangle BA, AH is equal to LK, KO, therefore the rectangle AB, EF is equal to the given rectangle NK, KO. and AB has to CD the given ratio of KL to KM; and from CD the given straight line GD being taken, the remainder CG has to EF the given ratio of KM to KN. Q. E. D.

PROB. II.

TO find three straight lines such, that the ratio of the first to the second is given; and if a given straight line be taken from the second, the ratio of the remainder to the third is given; also the sum of the squares of the first and third is given.

Let AB be the first straight line, BC the second, and BD the third. and because the ratio of AB to BC is given, and that if a given straight line be taken from BC, the ratio of the remainder to BD is given; therefore the excess of the first AB above a given straight line has a given ratio to the third BD. let AE be that given straight line, therefore the remainder EB has a given ratio to BD. let BD be placed at right angles to EB, and join DE, then the triangle EBD is given in species; wherefore the angle BED is given. let AE which is given in magnitude be given also in position, and the straight line ED will be given in position. join AD, and because the sum of the squares of AB, BD, that is, the square of AD is given, therefore the straight line AD is given in magnitude; and it is also given in position, because from the given point A it is drawn to the straight line ED given in position. therefore the point D in which the two straight lines AD, ED given in position cut one another is given. and the straight line DB which is at right angles to AB is given in position, and AB is given in position, therefore the point B is given. and the points A, D are given, wherefore the straight lines AB, BD are given. and the ratio of AB to BC is given, and therefore BC is given.

The Composition.

Let the given ratio of FG to GH be that which AB is required to have to BC, and let HK be the given straight line which is to be taken from BC, and let the ratio which the remainder. is required to have to BD be the given ratio of HG to GL, and place GL at right angles to FH, and join LF, LH. Next, as HG is to

GF, so make HK to AE; produce AE to N so that AN be the straight line to the square of which the sum of the squares of AB, BD is required to be equal; and make the angle NED equal to the angle GFL. from the center A at the distance AN describe a circle, and let its circumference meet ED in D, and draw DB perpendicular to AN, and DM making the angle BDM equal to the angle GLH. lastly, produce BM to C so that MC be equal to HK. then is AB the first, BC the second and BD the third of the straight lines that were to be found.

For the triangle EBD, FGL, as also DBM, LGH being equiangular, as EB to BD, so is FG to GL; and as DB to BM, so is LG to CH; therefore, ex acquali, as LB to BM, so is (FG to CH, and so is) AE to HK or MC; wherefore AB is to BC, as AE to HK, that is, as FG to GH, that is, in the given ratio. and from the straight line BC taking MC which is equal to the given straight line HK, the remainder BM has to BD the given ratio of HG to GL. and the form of the squares of AB, BD is equal to the square of AD or AN which is the given space. Q. E. D.

I believe it would be in vain to try to deduce the preceding Construction from an Algebraical Solution of the Problem.

FINIS.